Numerical Studies on High-order Fractional Laplacian

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Definitions of High-order Fractional Laplacian $(-\Delta)^s, s \in (1,2)$

- Pseudo-differential representation:

$$\mathcal{F}\left[(-\Delta)^s u\right](\xi) = \left[|\xi|^{2s}\mathcal{F}(u)\right], \quad s > 0,$$

- Hypersingular integral representation:

$$(-\Delta)^s u(\mathbf{x}) = \frac{1}{2(1-4^{1-s})}\frac{\Gamma(\frac{d}{2}+s)}{\pi^{\frac{d}{2}}\Gamma(-s)}\,\text{P.V.}\int_{\mathbb{R}^d}\frac{\delta_2 u(\mathbf{x},\mathbf{y})}{|\mathbf{y}|^{d+2s}}\,d\mathbf{y}, \quad s \in (1,2),$$

where

$$\delta_2 u(\mathbf{x},\mathbf{y}) = u(\mathbf{x}-2\mathbf{y}) - 4u(\mathbf{x}-\mathbf{y}) + 6u(\mathbf{x}) - 4u(\mathbf{x}+\mathbf{y}) + u(\mathbf{x}+2\mathbf{y}).$$

Motivation and Numerical Challenges

- **Application example:** Fractional viscoacoustic wave equation (Zhu and Harris, 2014)

$$\partial_t^2 u(\mathbf{x},t) = \alpha \underbrace{(-\Delta)^{\gamma+1}}_{\text{High-order}} u(\mathbf{x},t) + \beta \underbrace{(-\Delta)^{\gamma+\frac{1}{2}}}_{\text{Low-order}}\left[\partial_t u(\mathbf{x},t)\right],$$

  where $\gamma = \arctan(1/Q)/\pi \in (0, 0.5)$, Q is the quality factor.
- **Gap:** So far no numerical scheme for the fractional Laplacian $(-\Delta)^s$ with $s > 1$.
- **Main challenges:**
  - Nonlocality
  - Strong singularity
  - Rotational invariance
  - Storage cost
  - Computational cost
- **Goal of this study:** Develop the first numerical scheme for discretizing $(-\Delta)^s$.

Numerical Discretization of $(-\Delta)^s$

For the simplicity of notation, let's consider the 1D boundary value problem on $\Omega = (-L, L)$.

$$(-\Delta)^s u(x) = f, \quad x \in \Omega, \tag{1}$$

with extended Dirichlet boundary condition

$$u(x) = g(x), \quad x \in \Omega^c = \mathbb{R}\backslash\Omega. \tag{2}$$

First, rewrite the operator with $\xi = |y|$:

$$(-\Delta)^s u(x) = C_{1,s}\int_0^\infty \frac{u(x-2\xi)-4u(x-\xi)+6u(x)-4u(x+\xi)+u(x+2\xi)}{\xi^{1+2s}}\,d\xi$$

$$= C_{1,s}\int_0^\infty \underbrace{\frac{u(x-2\xi)-4u(x-\xi)+6u(x)-4u(x+\xi)+u(x+2\xi)}{\xi^4}}_{\Phi(x,\xi)}\xi^{3-2s}\,d\xi.$$

Then, denote $\xi_k = kh$, $h = L/K$, we have

$$(-\Delta)^s u(x) = C_{1,s}\int_0^\infty \Phi(x,\xi)\xi^{3-2s}\,d\xi = C_{1,s}\sum_{k=0}^{\infty}\int_{\xi_k}^{\xi_{k+1}}\Phi(x,\xi)\xi^{3-2s}\,d\xi,$$

where $\Phi(x,\xi)$ can be viewed as the central difference approximation to $u^{(4)}(x)$.

- For $k = 0$:

$$\int_{\xi_0}^{\xi_1}\Phi(x,\xi)\xi^{3-2s}\,d\xi \approx \Phi(x,h)\int_0^h \xi^{3-2s}\,d\xi = \frac{1}{p}h^p\Phi(x,h),$$

  where $p = 4 - 2s$.

- For $k > 0$:

$$\int_{\xi_k}^{\xi_{k+1}}\Phi(x,\xi)\xi^{3-2s}\,d\xi \approx \frac{1}{2}\Big(\Phi(x,\xi_k)+\Phi(x,\xi_{k+1})\Big)\int_{\xi_k}^{\xi_{k+1}}\xi^{3-2s}\,d\xi$$

$$= \frac{1}{2p}\Big(\xi_{k+1}^p - \xi_k^p\Big)\Big(\Phi(x,\xi_k)+\Phi(x,\xi_{k+1})\Big),$$

  i.e., weighted trapezoidal rule is used.

Denote x_i ($-K+1 \le i \le K-1$) as uniform grid points in Ω, and $u_i = u(x_i)$.

The discretized scheme for the high-order fractional Laplacian:

$$(-\Delta)_h^s u_i = \frac{C_{1,s}}{2ph^{2s}}\Bigg[6\Big(1 + 2^p + \sum_{k=2}^{\infty}\frac{(k+1)^p-(k-1)^p}{k^4}\Big)u_i$$

$$-4\big(1+2^p\big)u_{i\pm1} + \Big(1 + 2^p + (-4)\frac{3^p-1}{2^4}\Big)u_{i\pm2}$$

$$+\sum_{k\ge3}\Big((-4)\frac{(k+1)^p-(k-1)^p}{k^4} + \gamma(k)\frac{(\frac{k}{2}+1)^p-(\frac{k}{2}-1)^p}{(\frac{k}{2})^4}\Big)u_{i\pm k}\Bigg],$$

where $p = 4 - 2s$, and the coefficient

$$\gamma(k) = \begin{cases} 1, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.}\end{cases}$$

Remark:

- The coefficient matrix $\mathbf{A}$ is a Toeplitz matrix.
- The discretization can be generalized to high dimensional case.

Error Analysis

Denote the local truncation error as

$$e_h(\mathbf{x}) = (-\Delta)^s u(\mathbf{x}) - (-\Delta)_h^s u(\mathbf{x}), \quad \mathbf{x} \in \Omega.$$

Theorem 1 (Error estimates for $1 < s < 1.5$)

Let $(-\Delta)_h^s$ be a finite difference approximation of the high-order fractional Laplacian $(-\Delta)^s$, with h a small mesh size. For small $\varepsilon > 0$, there exists a constant $C > 0$ independent of h such that

1. if $u \in C^{2,2s-2+\varepsilon}(\mathbb{R})$, the local truncation error satisfies $\|e_h(\cdot)\|_\infty \le Ch^\varepsilon$.
2. if $u \in C^{4,2s-2+\varepsilon}(\mathbb{R})$, the local truncation error satisfies $\|e_h(\cdot)\|_\infty \le Ch^2$.

Theorem 2 (Error estimates for $1.5 \le s < 2$)

Let $(-\Delta)_h^s$ be the finite difference approximation to the operator $(-\Delta)^s$, with h a small mesh size. For small $\varepsilon > 0$, there exists a constant $C > 0$ independent of h, such that

1. if $u \in C^{3,2s-3+\varepsilon}(\mathbb{R})$, the local truncation error satisfies $\|e_h(\cdot)\|_\infty \le Ch^\varepsilon$.
2. if $u \in C^{5,2s-3+\varepsilon}(\mathbb{R})$, the local truncation error satisfies $\|e_h(\cdot)\|_\infty \le Ch^2$.

- **Case 1:** Consistency Take $u(x) = (1-x^2)^p$ with $p = 2s + \varepsilon$ and $\varepsilon = 3 - 2s$ for $s \in (1, 1.5)$, $\varepsilon = 4 - 2s$ for $s \in (1.5, 2)$.

s \ h		1/64	1/128	1/256	1/512	1/1024	1/2048
1.05		1.572E-2	9.619E-3	5.474E-3	3.018E-3	1.639E-3	8.843E-4
		c.r.	0.7806	0.8132	0.8591	0.8804	0.8906
1.25		1.394E-1	1.007E-1	7.206E-2	5.120E-2	3.628E-2	2.568E-2
		c.r.	0.4670	0.4851	0.4932	0.4968	0.4985
1.45		1.3937	1.2921	1.2016	1.1193	1.0435	0.9732
		c.r.	0.1092	0.1047	0.1024	0.1012	0.1006
1.5		5.631E-2	2.875E-2	1.441E-2	7.199E-3	3.596E-3	1.797E-3
		c.r.	0.9697	0.9965	1.001	1.001	1.001
1.75		1.2685	8.991E-1	6.307E-1	4.430E-1	3.119E-1	2.200E-1
		c.r.	0.4965	0.5116	0.5098	0.5061	0.5034
1.95		9.7483	9.2094	8.5794	7.9824	7.4335	6.9281
		c.r.	0.082	0.1022	0.1040	0.1028	0.1016

Operator error in $\|e_h\|_\infty$ under consistency condition has $\mathcal{O}(h^\varepsilon)$ accuracy.

- **Case 2:** 2nd-order Accuracy Take $u(x) = (1-x^2)_+^{4+\frac{\alpha}{2}}$.

s \ h	1/8	1/16	1/32	1/64	1/128
1.05	2.0795e-01	5.8767e-02	1.5157e-02	3.8206e-03	9.5724e-04
	c.r.	1.8232	1.9550	1.9881	1.9968
1.5	1.3368e+00	1.9829e-01	2.5664e-02	5.1980e-03	1.2908e-03
	c.r.	2.7531	2.9498	2.3037	2.0097
1.95	2.6197e+01	8.0881e+00	2.0973e+00	8.1771e-01	2.4300e-01
	c.r.	1.6956	1.9473	1.3589	1.7506

Operator errors in $\|e_h\|_\infty$, get $\mathcal{O}(h^2)$ accuracy.

Boundary Value Problems

For the boundary value problems, we obtain the linear system

$$A\mathbf{U} + \mathbf{b} = \mathbf{F},$$

where $\mathbf{U}$ denotes the numerical solution, A denote the coefficient matrix of x, and $\mathbf{F}$ represents the right-hand side of the equation (1).

Remark: Here $\mathbf{b}$ comes from the boundary condition, that is, comes from those terms $u_{i\pm k}$ with $|i \pm k| \ge K$. For homogeneous boundary condition, $\mathbf{b} = \mathbf{0}$.

- **Case 3:** B.V.P. with compact support solution: Consider

$$f(x) = \frac{2^{2s}\Gamma(s+\frac{1}{2})\Gamma(7)}{\sqrt{\pi}\Gamma(7-s)}\,{}_2F_1\Big(s+\frac{1}{2}, s-6; \frac{1}{2}; x\Big), \quad x \in (-1,1),$$

$$g(x) = 0, \quad x \in \mathbb{R}\backslash(-1,1),$$

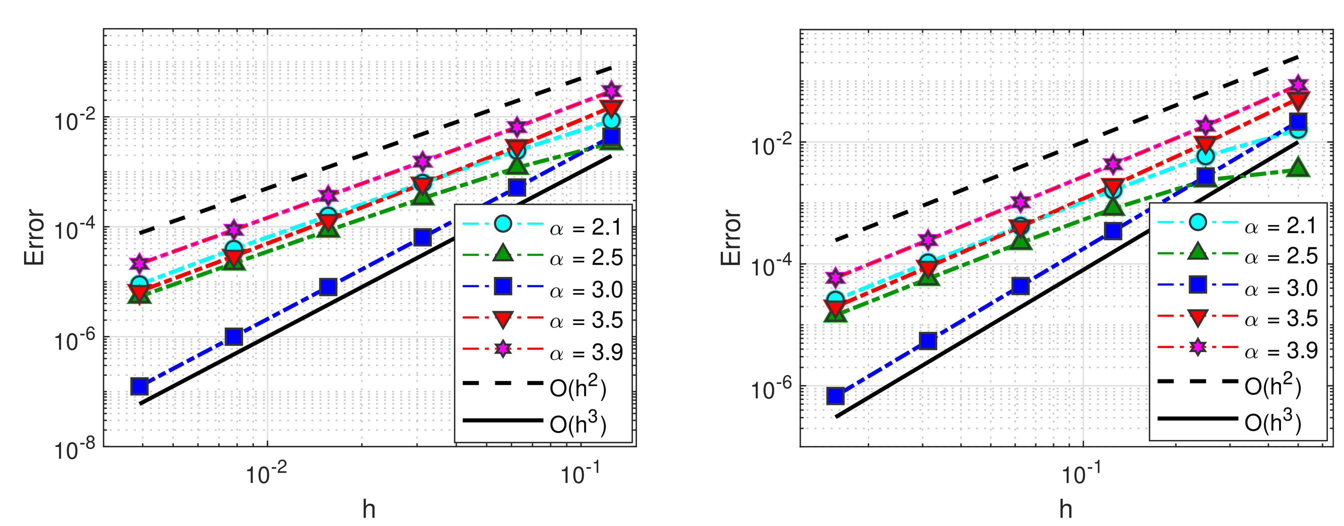
  with exact solution $u(x) = (1-x^2)_+^6$. Where ${}_2F_1$ denotes the Gauss hypergeometric function.

- **Case 4:** B.V.P. with global solution: Consider

$$f(x) = \frac{2^{2s}\Gamma(s+\frac{1}{2})}{\sqrt{\pi}}\,{}_1F_1\Big(s+\frac{1}{2}; \frac{1}{2}; -x^2\Big), \quad x \in (-4,4),$$

$$g(x) = e^{-x^2}, \quad x \in \mathbb{R}\backslash(-4,4),$$

  with exact solution $u(x) = \exp(-x^2)$. Where ${}_1F_1$ represents the confluent hypergeometric function.



(a) Errors for Case 3 (b) Errors for Case 4

Numerical errors for Boundary value problems

References

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