A Monotone Meshfree Finite Difference Method for Linear Elliptic PDEs via Nonlocal Relaxation

Qihao Ye and Xiaochuan Tian

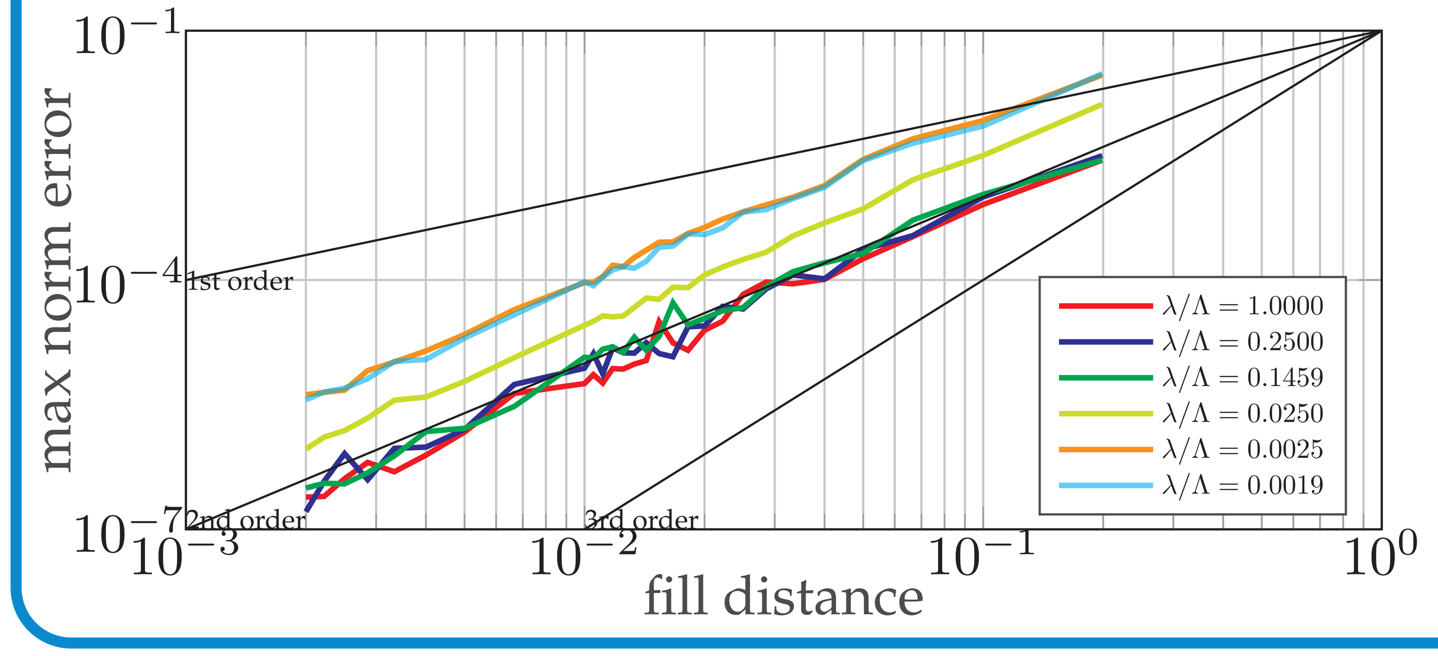
University of California, San Diego (UCSD)

q8ye@ucsd.edu   xctian@ucsd.edu

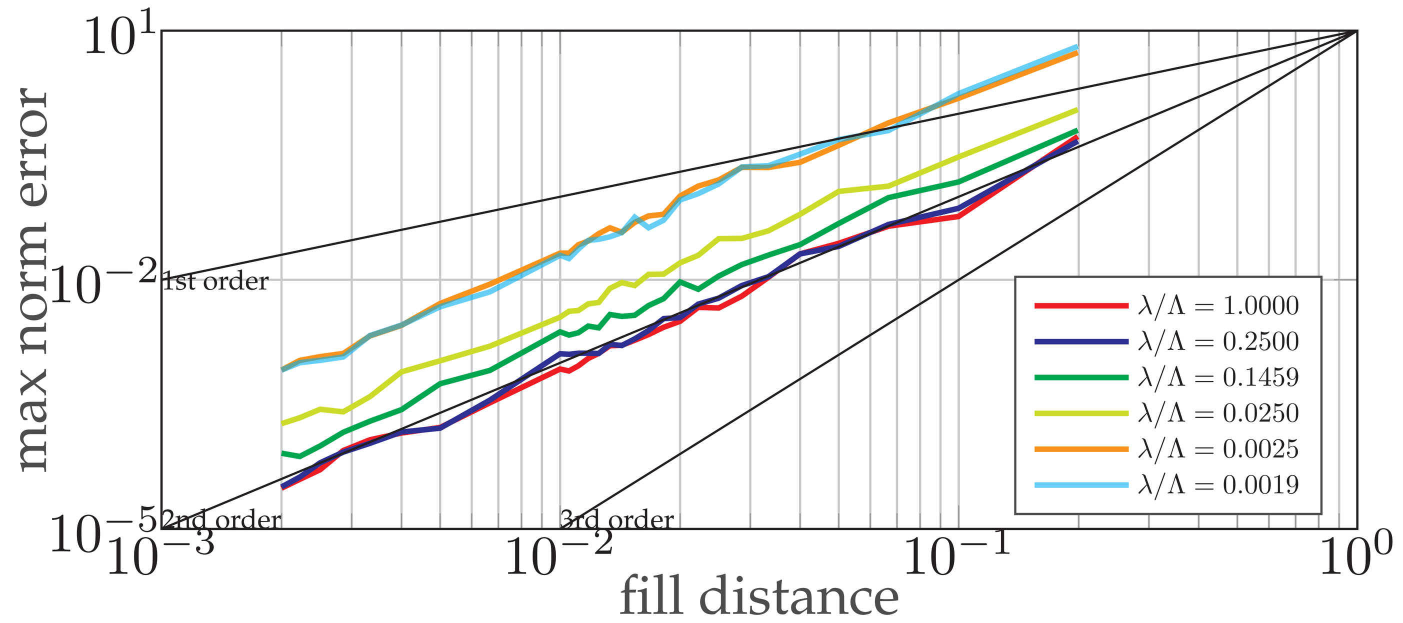
UC San Diego

Numerical Results

Unit Disk Domain, $d = 2, p = 2$,
$u(x_1, x_2) = x_1 x_2 + \cos(x_1)\exp(x_2)$



Unit Disk Domain, $d = 2, p = 2$,
$u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$



SCAN to see the **implementation** and **more results**

Basic Ideas

MAIN GOAL: Solve the second-order linear elliptic equations in non-divergence form

$$\begin{cases} -Lu(\boldsymbol{x}) := -\sum_{i,j=1}^{d} a^{ij}(\boldsymbol{x})\partial_{ij}u(\boldsymbol{x}) = f(\boldsymbol{x}) & \boldsymbol{x}\in\Omega \\ u(\boldsymbol{x}) = g(\boldsymbol{x}) & \boldsymbol{x}\in\partial\Omega \end{cases},$$

for an open bounded domain $\Omega \in \mathbb{R}^d$. The matrix $A(\boldsymbol{x}) = (a^{ij}(\boldsymbol{x}))_{i,j=1}^d$ is assumed to be symmetric and positive definite satisfying the uniform ellipticity condition

$$\lambda|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi}^T A(\boldsymbol{x})\boldsymbol{\xi} \le \Lambda|\boldsymbol{\xi}|^2 \qquad \forall \boldsymbol{\xi}\in\mathbb{R}^d$$

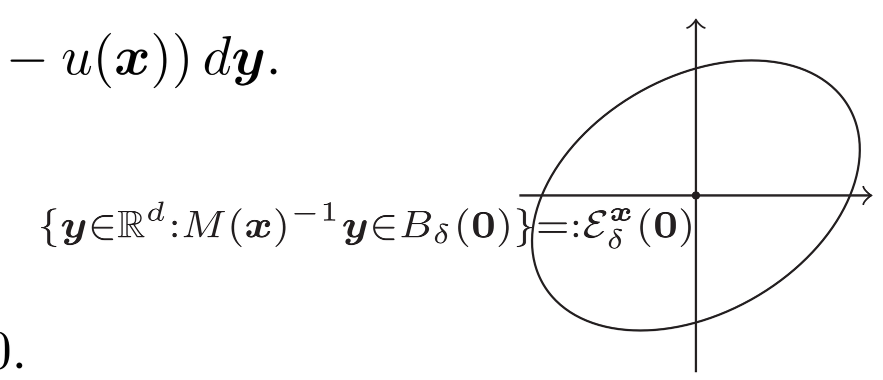
for positive constants λ, Λ with ratio $\lambda/\Lambda \le 1$.
Denote $M(\boldsymbol{x}) := (A(\boldsymbol{x}))^{1/2}$.

NONLOCAL RELAXATION METHOD: The nonlocal elliptic operator[2, 3, 4] can be defined as

$$\mathcal{L}_\delta u(\boldsymbol{x}) = \int_{B_\delta(\mathbf{0})} \frac{1}{\delta^{d+2}}\gamma\left(\frac{|\boldsymbol{z}|}{\delta}\right)(u(\boldsymbol{x}+M(\boldsymbol{x})\boldsymbol{z}) - u(\boldsymbol{x}))\,d\boldsymbol{z}$$

$$= \int_{\mathcal{E}_\delta^{\boldsymbol{x}}(\mathbf{0})} \frac{1}{\delta^{d+2}}\gamma\left(\frac{|M(\boldsymbol{x})^{-1}\boldsymbol{y}|}{\delta}\right)\det(M(\boldsymbol{x}))^{-1}(u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}))\,d\boldsymbol{y}$$

$$:= \int_{\mathcal{E}_\delta^{\boldsymbol{x}}(\mathbf{0})} \rho_\delta(\boldsymbol{x},\boldsymbol{y})(u(\boldsymbol{x}+\boldsymbol{y}) - u(\boldsymbol{x}))\,d\boldsymbol{y}.$$

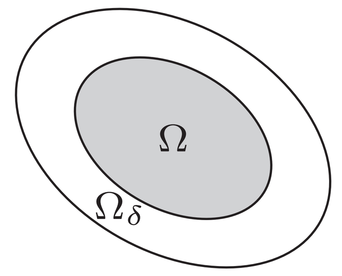


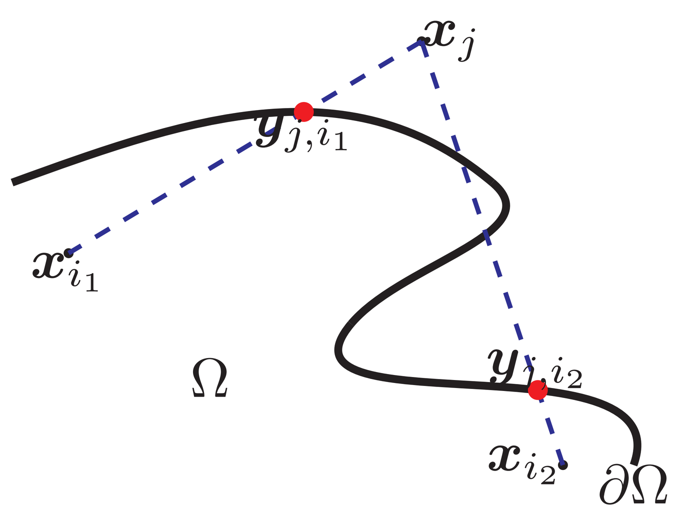
It can be shown that

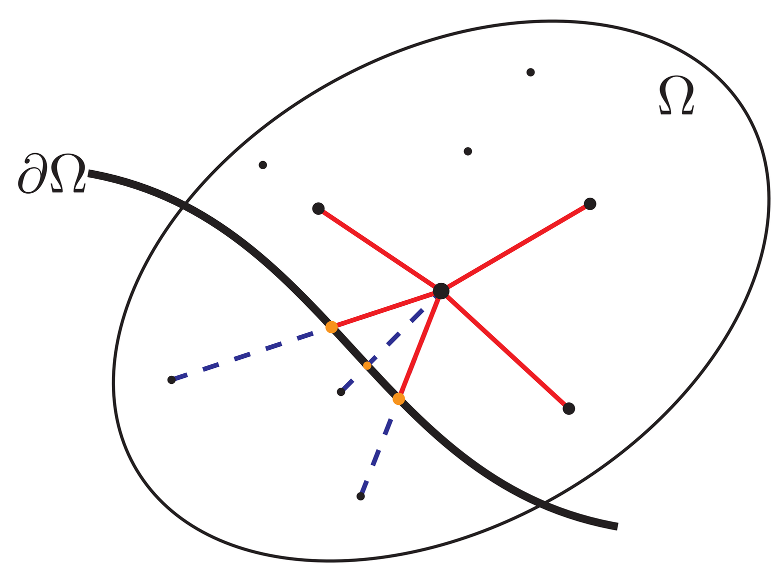
$$\mathcal{L}_\delta u(\boldsymbol{x}) \to Lu(\boldsymbol{x}) \quad \text{as} \quad \delta \to 0.$$

BOUNDARY TREATMENT: Point cloud contains

1. Interior points $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ (in Ω),
2. Boundary points $\{\boldsymbol{x}_{N+1}, \ldots, \boldsymbol{x}_M\}$ (in $\Omega_\delta \setminus \Omega$).







OPTIMIZATION BASED MESHFREE METHOD: We use the following minimization problem[1, 6] to select a stencil for interior point $\boldsymbol{x}_i$ and p the order of the polynomial space:

$$\{\beta_{j,i}\} = \underset{\{\beta_{j,i}\}\in\overline{S}_{\delta,h,p}}{\arg\min} \sum_j \frac{\beta_{j,i}}{\rho_\delta(\boldsymbol{x}_i, \boldsymbol{y}_{j,i} - \boldsymbol{x}_i)},$$

where h is the fill distance[5] and

$$\boldsymbol{y}_{j,i} = \begin{cases} \boldsymbol{x}_j & ,\boldsymbol{x}_j \in \overline{\Omega} \\ \text{projection from } \boldsymbol{x}_j \text{ to } \boldsymbol{x}_i \text{ at } \partial\Omega & ,\boldsymbol{x}_j \in \Omega_\delta \setminus \overline{\Omega} \end{cases},$$

$$\overline{S}_{\delta,h,p} := \left\{\{\beta_{j,i}\} : \beta_{j,i} \ge 0 \text{ and } \mathcal{L}_{\delta,\Omega}^h u(\boldsymbol{x}_i) = \mathcal{L}_\delta u(\boldsymbol{x}_i)\ \forall u \in \mathcal{P}_p(\mathbb{R}^d)\right\},$$

$$\mathcal{L}_{\delta,\Omega}^h u(\boldsymbol{x}_i) = \sum_{\boldsymbol{x}_j \in \mathcal{E}_\delta^{\boldsymbol{x}_i}(\boldsymbol{x}_i)} \beta_{j,i}(u(\boldsymbol{y}_{j,i}) - u(\boldsymbol{x}_i)).$$

Analysis(Ye-Tian, 2022)

Lemma 1: Assume $\overline{S}_{\delta,h,p}$ is not empty and $C > 0$ is a generic constant.

1. If $p \ge 2$ and $u \in C^2(\overline{\Omega})$, then $|\mathcal{L}_{\delta,\Omega}^h u(\boldsymbol{x}_i) - Lu(\boldsymbol{x}_i)| \to 0$ as $\delta \to 0$ for all $\boldsymbol{x}_i \in \Omega$.
2. If $p \ge 2$ and $u \in C^{2,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1]$, then $|\mathcal{L}_{\delta,\Omega}^h u(\boldsymbol{x}_i) - Lu(\boldsymbol{x}_i)| \le C|u|_{C^{2,\alpha}(\overline{\Omega})}\delta^\alpha$ for all $\boldsymbol{x}_i \in \Omega$.
3. If $p \ge 3$ and $u \in C^{3,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1]$, then $|\mathcal{L}_{\delta,\Omega}^h u(\boldsymbol{x}_i) - Lu(\boldsymbol{x}_i)| \le C|u|_{C^{3,\alpha}(\overline{\Omega})}\delta^{1+\alpha}$ for all $\boldsymbol{x}_i \in \Omega$.

Theorem 2: In $d = 2$, there exists a constant $C > 0$ such that if

$$h \le C\delta\sqrt{\lambda/\Lambda}$$

then $\overline{S}_{\delta,h,2}$ is not empty.

Theorem 3: In $d = 2$, assume $\overline{S}_{\delta,h,p}$ is not empty, let u be the real solution and u_δ^h be the solution solved by the discrete operator and $C > 0$ is a generic constant.

1. If $p \ge 2$ and $u \in C^{2,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1]$, then
$$\max_{\boldsymbol{x}_i\in\Omega}\left|u(\boldsymbol{x}_i) - u_\delta^h(\boldsymbol{x}_i)\right| \le C|u|_{C^{2,\alpha}(\overline{\Omega})}\left(\sqrt{\lambda/\Lambda}\right)^{-\alpha} h^\alpha$$
2. If $p \ge 3$ and $u \in C^{3,\alpha}(\overline{\Omega})$ for $\alpha \in (0,1]$, then
$$\max_{\boldsymbol{x}_i\in\Omega}\left|u(\boldsymbol{x}_i) - u_\delta^h(\boldsymbol{x}_i)\right| \le C|u|_{C^{3,\alpha}(\overline{\Omega})}\left(\sqrt{\lambda/\Lambda}\right)^{-(1+\alpha)} h^{1+\alpha}$$

References

[1] B. Seibold, "Minimal positive stencils in meshfree finite difference methods for the poisson equation," *Computer Methods in Applied Mechanics and Engineering*, vol. 198, no. 3-4, pp. 592–601, 2008.

[2] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, "Analysis and approximation of nonlocal diffusion problems with volume constraints," *SIAM review*, vol. 54, no. 4, pp. 667–696, 2012.

[3] S. A. Silling, "Reformulation of elasticity theory for discontinuities and long-range forces," *Journal of the Mechanics and Physics of Solids*, vol. 48, no. 1, pp. 175–209, 2000.

[4] R. H. Nochetto and W. Zhang, "Discrete abp estimate and convergence rates for linear elliptic equations in non-divergence form," *Foundations of Computational Mathematics*, vol. 18, no. 3, pp. 537–593, 2018.

[5] H. Wendland, *Scattered data approximation*. Cambridge university press, 2004, vol. 17.

[6] N. Trask, H. You, Y. Yu, and M. L. Parks, "An asymptotically compatible meshfree quadrature rule for nonlocal problems with applications to peridynamics," *Computer Methods in Applied Mechanics and Engineering*, vol. 343, pp. 151–165, 2019.

Research supported in part by NSF DMS-2111608.