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An Introduction to the Malliavin-Stein Method

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Luxembourg → Zoom → Banff
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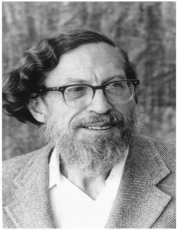
Charles Stein and Paul Malliavin

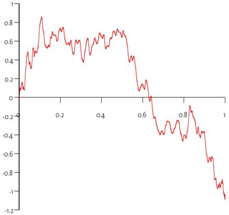


In 2009, together with I. Nourdin, we discovered a way of combining **Stein's method for probabilistic approximations** (Stein, 1972) ...

... with the **Malliavin calculus of variations** on a Gaussian space (Malliavin, 1978).

Crucial notion: **integration by parts formulae**

Where it All Started



Initial motivation: quantitative fluctuations of functionals of **infinite-dimensional Gaussian fields**, like e.g. a **(fractional) Brownian motion** $\{X_t\}$.

Key notion: **Breuer-Major CLTs**.

Typical examples:

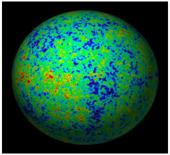
- ⋆ **Power variations**:

$$\sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p, \quad n \to \infty;$$

- ⋆ **Centered empirical moments**:

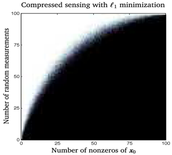
$$\int_0^T \left(X_t - \int_0^T X_u du \right)^m dt, \quad T \to \infty.$$

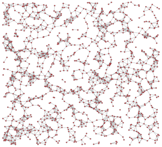
Distinguished Applications/Extension



Level/excursion sets of **random fields on manifolds** (*Marinucci and Peccati, 2011, Nourdin, Peccati & Rossi, 2017, ...*)

Phase transitions in **sparse recovery problems** (*Goldstein, Nourdin & Peccati, 2014*).





Random geometric graphs (*Reitzner & Schulte, 2010, Last, Peccati & Schulte, 2016, Lachièze-Rey, Peccati & Yang, 2022, Schulte & Yukich, 2021, ...*)

Setting, I

⋆ Main focus on **normal approximations**, with usual notation:

- $\mathscr{N}(\mu, \sigma^2)$ (1-dimensional)
- $\mathscr{N}_d(\mathbf{a}, C)$ (d-dimensional).

⋆ For $m \geq 1$,

$$\mathbf{g}_m = (g_1, ..., g_m)$$

indicates a vector of i.i.d. $\mathscr{N}(0, 1)$ random variables.

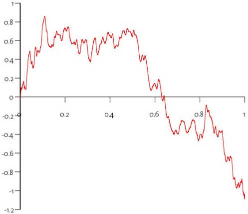
SETTING, II

⋆ We write

$$W = \{W_t : t \in [0,1]\}$$

for a **standard Brownian motion** on $[0,1]$:

- W is Gaussian,
- $W_0 = 0$,
- $\mathbb{E}[W_t] = 0$,
- $\mathbf{Cov}(W_s, W_s) = s \wedge t$,
- W is continuous.



⋆ For all $h \in L^2([0,1])$ (deterministic)

$$W(h) := \int_0^1 h(s)dW_s \sim \mathcal{N}(0, \|h\|^2) \quad \text{(jointly Gaussian)}$$

The Problem, I

- ⋆ Consider a square integrable random variable $F = F(W)$ such that $\mathbb{E}[F] = 0$, $\mathbb{E}F^2 = 1$.

- ⋆ **Goal**: compare the distribution of F and that of

$$Z \sim \mathcal{N}(0,1).$$

- ⋆ **Tool**: the **1-Wasserstein distance**:

$$\mathbf{W}_1(F,Z) := \inf_{A\sim F, B\sim Z} \mathbb{E}|A-B| = \sup_{h\in \mathrm{Lip}(1)} |\mathbb{E}h(F) - \mathbb{E}h(Z)|.$$

- ⋆ **Remark**: the analysis extends to the *Kolmogorov*, *total variation*, *bounded Wasserstein* (Fortet-Mourier) (...) distances.

The Problem, II

- ⋆ For a smooth $g : \mathbb{R} \to \mathbb{R}$, introduce the operator

$$\mathcal{T}g(x) := xg(x) - g'(x)$$

(adjoint of $g \mapsto g'$ in $L^2(\mathbb{R}, e^{-x^2/2}/\sqrt{2\pi})$).

- ⋆ <u>**Stein's method**</u>: $\mathbf{W}_1(F, Z)$ is actually bounded by a **discrepancy**:

$$\mathbf{W}_1(F, Z) \leq \mathcal{S}(F, \mathcal{T}, \mathcal{G}) := \sup_{g \in \mathcal{G}} |\mathbb{E}[\mathcal{T}g(F)]|,$$

where $\mathcal{G} := \{g \in C^1 : \|g'\| \leq \sqrt{2/\pi},\ \|g''\| \leq 2\}$.

The Problem, III

⋆ Fix $g \in \mathcal{G}$: *how to (uniformly) bound*

$$\left|\mathbb{E}[\mathcal{T}g(F)]\right| = \left|\mathbb{E}[Fg(F)] - \mathbb{E}[g'(F)]\right| \quad ?$$

⋆ **Idea**: Assume that $F = F(W)$ belongs to the domain of some **Malliavin-type operators**.

The Ornstein-Uhlenbeck Semigroup

⋆ For $t \geq 0$ and $F = F(W)$ integrable, set

$$P_tF = P_tF(W) := \mathbb{E}\left[F(e^{-t}W + \sqrt{1-e^{-2t}}W') \,\middle|\, W\right],$$

where W' = independent copy of W.

⋆ $\{P_t : t \geq 0\}$ = "**Ornstein-Uhlenbeck semigroup**" (Mehler's form).

⋆ One has that

$$P_0F = F \quad \text{and} \quad P_\infty F = \mathbb{E}[F].$$

SOME FACTS, I

⋆ For $n \geq 1$ and a symmetric (deterministic) $f \in L^2([0,1]^n)$, define the **Wiener-Itô multiple stochastic integral** of order n:

$$I_n(f) := n! \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} f(t_1, ..., t_n) dW_{t_n} dW_{t_{n-1}} \cdots dW_{t_1}.$$

⋆ For $t > 0$, the **eigenspaces** of $P_t : L^2(\sigma(W)) \to L^2(\sigma(W))$ are the spaces $\{C_n : n \geq 0\}$ defined as: $C_0 := \mathbb{R}$, and

$$C_n := \{ I_n(f) : f \in L^2([0,1]^n), \text{symmetric} \}, \quad n \geq 1.$$

⋆ C_n := "n**th Wiener Chaos of** W" ($\simeq$ *infinite-dimensional counterpart of Hermite polynomials of degree* n)

Some Facts, II

- For every $n \geq 1$ and $t \geq 0$,

$$P_t I_n(f) = e^{-nt} I_n(f), \quad \forall f \in L^2([0,1]^n).$$

- **[Wiener Chaos Expansion]** for every $F \in L^2(\sigma(W))$, $\exists!$ $\{f_n : n \geq 1\}$ such that

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) \quad \text{(in } L^2\text{)}.$$

- As a consequence, for every $F \in L^2(\sigma(W))$,

$$P_t F = \mathbb{E}[F] + \sum_{n=1}^{\infty} e^{-nt} I_n(f_n).$$

- Using **Itô's isometry**,

$$\mathbb{E}F^2 = \mathbb{E}^2 F + \sum_{n=1}^{\infty} n! \|f_n\|^2.$$

Some Facts, III

⋆ The **generator** L of $\{P_t\}$ is given by

$$LF = -\sum_{n=1}^{\infty} n I_n(f_n), \quad F \in \operatorname{dom} L.$$

⋆ The **pseudo-inverse** L^{-1} of L is given by: for all $F \in L^2(\sigma(W))$

$$L^{-1}F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n)$$

⋆ One has that

$$LL^{-1}F = L^{-1}LF = F - \mathbb{E}F.$$

Malliavin Derivatives, I

- For $F = f(W_{t_1}, ..., W_{t_d})$, ($f$ smooth) define the **Malliavin derivative** of F as

$$D_x F := \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f(W_{t_1}, ..., W_{t_d}) \mathbf{1}_{[0,t_i]}(x), \quad x \in [0,1].$$

- The random element DF takes values in $L^2([0,1])$.
- By density and closability, the definition of D can be extended to the class

$$\mathbb{D}^{1,2} := \left\{ F : \sum_n n n! \|f_n\|^2 < \infty \right\},$$

in which case

$$D_x F = \sum_n n I_{n-1}(f_n(x, \cdot)).$$

Malliavin Derivatives, II

- **Chain Rule**: for φ smooth

$$D\varphi(F_1, \ldots, F_m) = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \varphi(F_1, \ldots, F_m) DF_i.$$

- Write δ for the **adjoint** of D (the "**Skorohod integral**"). It verifies: for all $u \in \operatorname{dom} \delta$ and all $F \in \mathbb{D}^{1,2}$,

$$\mathbb{E}[F\delta(u)] = \mathbb{E}\left[\int_0^1 u(x) D_x F \, dx\right] := \mathbb{E}\langle DF, u\rangle$$

  (**"integration by parts"**).

- Key relation: $F \in \operatorname{dom} L$ if $F \in \mathbb{D}^{1,2}$ and $DF \in \operatorname{dom} \delta$, in which case

$$LF = -\delta DF.$$

REMARKS

⋆ When W is replaced by $\mathbf{g}_m = (g_1, ..., g_m) \sim \mathcal{N}_m(\mathbf{0}, \mathbf{I}_m)$, Malliavin operators boil down to familiar objects:

- $Df(\mathbf{g}_m) = \nabla f(\mathbf{g}_m)$;
- $\delta(f_1(\mathbf{g}_m), ..., f_m(\mathbf{g}_m)) = \sum_{i=1}^m g_i f_i(\mathbf{g}_m) - \sum_{i,j} \frac{\partial}{\partial x_j} f_i(\mathbf{g}_m)$;
- $L = -\delta\nabla$ is a second-order differential operator;
- $\delta = \mathcal{T}$ for $m = 1$.

⋆ In general, for F, G sufficiently smooth,

$$\langle DF, DG \rangle = \frac{1}{2}\left[L(FG) - F\,LG - G\,LF\right] := \textbf{“Carré du champ”}$$

See: *Ledoux, 2012*; *Azmoodeh, Campese & Poly, 2013*; *Nourdin, Peccati & Swan, 2014*; *Nourdin, Ledoux & Peccati, 2016*.

Crucial Computation

- ⋆ For $F = F(W)$ such that $\mathbb{E}F = 0$ and $\mathbb{E}F^2 = 1$, we want to bound
$$\left|\mathbb{E}[\mathcal{T}g(F)]\right| = \left|\mathbb{E}[Fg(F)] - \mathbb{E}[g'(F)]\right|,$$
for all g such that $|g'| \leq \sqrt{2/\pi}$.
- ⋆ Assume $F \in \mathbb{D}^{1,2}$. Then:

$$\begin{aligned}\mathbb{E}[Fg(F)] &= \mathbb{E}[LL^{-1}F\,g(F)] = -\mathbb{E}[\delta(DL^{-1}F)\,g(F)] \\ &= -\mathbb{E}\langle Dg(F), DL^{-1}F\rangle = \mathbb{E}[g'(F)\langle DF, -DL^{-1}F\rangle].\end{aligned}$$

- ⋆ Finally, writing $H_F := \langle DF, -DL^{-1}F\rangle$

$$\sqrt{\frac{\pi}{2}}\left|\mathbb{E}[\mathcal{T}g(F)]\right| \leq \mathbb{E}\,|1 - H_F| \leq \mathbf{Var}^{1/2}(H_F).$$

Basic Bound

Let $Z \sim \mathcal{N}(0,1)$.

Theorem (Nourdin & Peccati, 2009)
Let $F = F(W) \in \mathbb{D}^{1,2}$ *be such that* $\mathbb{E}F = 0$ *and* $\mathbb{E}F^2 = 1$. *Then,*

$$\mathbf{W}_1(F,Z) \leq \sqrt{\frac{2}{\pi}}\,\mathbf{Var}^{1/2}(\langle DF, -DL^{-1}F\rangle).$$

Fourth Moment Theorem

Let $Z \sim \mathcal{N}(0,1)$.

Theorem (Nourdin, Peccati & Reinert, 2010)
For $q = 2, 3, \ldots$*, assume that* $F \in C_q$ *has variance one. Then,*

$$\mathbf{W}_1(F,Z) \leq \sqrt{\frac{2q-2}{3\pi q}(\mathbb{E}F^4 - \mathbb{E}Z^4)} \left(= \sqrt{\frac{2q-2}{3\pi q}(\mathbb{E}F^4 - 3)} \right).$$

Remark: recovers *Nualart & Peccati, 2005.*

Second Order Inequalities

- ⋆ The relation

$$\mathbb{E}[Fg(F)] = \mathbb{E}[g'(F)\langle DF, -DL^{-1}F\rangle],$$

is also the crucial identity leading to **second order Poincaré inequalities** (*Chatterjee, 2007*, *Nourdin, Peccati & Reinert, 2010*).

- ⋆ In our setting, such a result reads : for a smooth F,

$$\mathbf{W}_1(F,Z) \lesssim \mathbb{E}[\|D^2F\|_{op}^4]^{1/4}\mathbb{E}[\|DF\|^4]^{1/4}.$$

- ⋆ Compare with the usual **Poincaré inequality**:

$$\mathbf{Var}(F) \leq \mathbb{E}\|DF\|^2.$$

Multidimensional Extensions

- ⋆ Multidimensional bounds in the **1-Wasserstein distance**: *Nourdin, Peccati and Réveillac, 2008*. In the convex distance: *Nourdin, Peccati & Yang, 2021*.
- ⋆ Bounds on **relative entropy** (any dimension): *Nourdin, Peccati & Swan, 2014*.
- ⋆ Application to **functional inequalities** (entropy and transport): *Ledoux, Nourdin and Peccati, 2016*
- ⋆ Characterization of **convergence on Wiener chaos**: *Nourdin and Poly, 2014*, and *Nourdin, Nualart & Peccati, 2015*.

Poisson Measures

- ⋆ Let $(A, \mathscr{A}, \mu)$ be a Polish space endowed with a locally finite Borel measure μ.

- ⋆ We denote by η a **Poisson measure** with **intensity** μ. Recall that: (i) $\eta(B) \sim \text{Po}(\mu(B))$, and (ii) $\forall B, C \in \mathscr{A}$ s.t. $B \cap C = \emptyset$, $\eta(B)$ and $\eta(C)$ are independent.

- ⋆ Standard arguments yield that η is indeed a **random point measure** such that

$$\mathbb{P}\Big\{\eta(\{x\}) \in \{0,1\},\ \forall x \in A\Big\} = 1.$$

- ⋆ Here, the role of D is played by the **"add-one cost operator"**

$$D^+F(\eta) = F(\eta + \delta_x) - F(\eta),$$

(**NB**: this is not a derivation).

Typical Statements

- *Peccati, Solé, Utzet & Taqqu, 2010*: for $Z \sim \mathcal{N}(0,1)$ and F "regular" and such that $\mathbb{E}F = 0$, $\mathbb{E}F^2 = 1$

$$\mathbf{W}_1(F,Z) \lesssim \sqrt{\mathbf{Var}(X_F)} + \left(\int_Z (D_x^+ F)^4 \mu(dx) \right)^{1/2},$$

  where $X_F := -\int_A D_x^+ F \, (D_x^+ L^{-1} F) \, \mu(dx)$.

- **Second order Poincaré inequalities** are available also in this framework (*Last, Peccati & Schulte, 2016*): for Z, F as before,

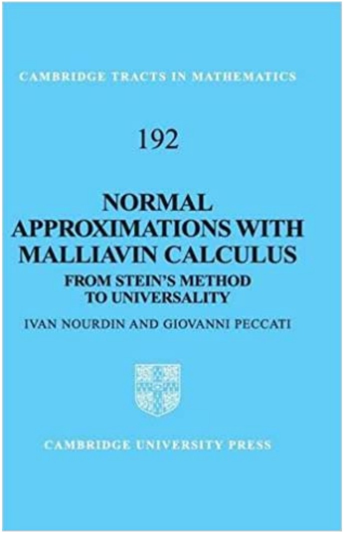
$$\mathbf{W}_1(F,Z)^2 \lesssim \mathbb{E}\left[\int (D_x^+ F)^4 \, \mu(dx) \right] + \mathbb{E}\left[\int (D_x^+ F)^2 \, \mu(dx) \right] \times \mathbb{E}\left[\int \int (D_x^+ D_y^+ F)^2 \, \mu(dx)\mu(dy) \right],$$

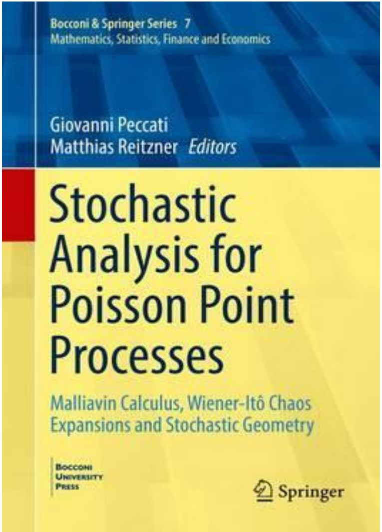
  yielding that normality arises from "small local contributions", and "vanishing second order interactions".

Pot-Pourri

- ⋆ **Fourth moment theorems on the Poisson space**: *Döbler & Peccati, 2018*; *Döbler, Vidotto & Zheng, 2019*.
- ⋆ **Second-order inequalities and "geometric stabilization"**: *Lachièze-Rey, Schulte & Yukich, 2017*; *Schulte & Yukich, 2018-2021* (multidimensional convex distance).
- ⋆ **Geometric stabilization without Poincaré**: *Lachièze-Rey, Peccati & Yang, 2022*
- ⋆ **Stable convergence on the Poisson space**: *Herry, 2021*.

Two Books (2012 & 2016)





A Webpage

https://sites.google.com/site/malliavinstein

Malliavin-Stein approach

A webpage maintained by Ivan Nourdin

Why this webpage?

- In a seminal paper of 2005, Nualart and Peccati discovered a surprising central limit theorem (called the ``*fourth moment theorem*'' in the sequel; alternative proofs can be found here, here and here) for sequences of multiple stochastic integrals of a fixed order: in this context, convergence in distribution to the standard normal law is actually equivalent to convergence of just the fourth moment! Shortly afterwards, Peccati and Tudor gave a multidimensional version of this characterization.
- Since the publication of these two pathbreaking papers, many improvements and developments on this theme have been considered. Among them is the work by Nualart and Ortiz-Latorre, giving a new proof only based on Malliavin calculus and the use of integration by parts on Wiener space. A second step is my joint paper ``Stein's method on Wiener chaos'' written in collaboration with Peccati in which, by bringing together Stein's method with Malliavin calculus, we were able (among other things) to associate quantitative bounds to the *fourth moment theorem*.
- It turns out that Stein's method and Malliavin calculus fit together admirably well, and that their interaction has led to some remarkable new results involving central and non-central limit theorems for functionals of infinite-dimensional Gaussian fields.

Final Words

THANK YOU!