p-weak differentiable structure on metric spaces

Sylvester Eriksson-Bique, Elefterios Soultanis

SUOMEN AKATEMIA

JENNY JA ANTTI WIHURIN RAHASTO

JYVÄSKYLÄN YLIOPISTO

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Questions:

Question

Is there a natural linear structure on $W^{1,p}(X)$?

$$f(\gamma_1) - f(\gamma_0) = \int_\gamma df \cdot ds.$$

Question

Is $g_f = |df|$ for some linear and pointwise defined df?

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_\gamma g_f \, ds.$$

How to make sense in manifolds?

Definition

A chart $(U, \phi : U \to \mathbb{R}^n)$ on a manifold consists of a diffeomorphism ϕ and $U \subset M$ open.

Definition

If $f : M \to \mathbb{R}$ is Lipschitz, then any chart $(U, \phi : U \to \mathbb{R}^n)$ and a.e. $x \in U$, there exists $df_x \in (\mathbb{R}^n)^*$

$$f(y) - f(x) = df_x(\phi(y) - \phi(x)) + o(d(x,y)).$$

$$f(\gamma_1) - f(\gamma_0) = \int_0^1 df_x((\phi \circ \gamma)'_t)dt.$$

$$g_f = \sup_{|v|=1} |df(v)|.$$

How to make sense in PI-spaces?

Definition

Chart: $U \subset X, \phi : X \to \mathbb{R}^n$ Lipschitz. For every $f : X \to \mathbb{R}$ there exists a unique: $df_x \in (\mathbb{R}^n)^*$

$$f(y) - f(x) = df_x(\phi(y) - \phi(x)) + o(d(x,y)).$$

Theorem

(Cheeger '99) If (X, d, μ) is measure doubling and satisfies a Poincaré-inequality, then there exist charts $(U_i, \phi_i : X \to \mathbb{R}^{n_i})$ with

1. $\sup_{i\in\mathbb{N}} n_i \leq C(D, C_{PI})$.
2. $\mu(X \setminus \bigcup U_i) = 0$.

$$f(\gamma_1) - f(\gamma_0) = \int_0^1 \sum_{i\in\mathbb{N}} df_x((\phi_i \circ \gamma)'_t) 1_{U_i}(\gamma_t) dt.$$

$$g_f = |df|.$$

What about other X?

Only consider points along curves for a.e. curve: $x = \gamma_s, y = \gamma_t$.

Definition

p-**weak chart**: $U \subset X, \phi : X \to \mathbb{R}^n$ Lipschitz. For every $f : X \to \mathbb{R}$ there exists a Unique: $df_x \in (\mathbb{R}^n)^*$

$$f(\gamma_s) - f(\gamma_t) = df_x(\phi(\gamma_s) - \phi(\gamma_t)) + o(d(\gamma_s, \gamma_t)),$$

for p.a.e γ and a.e. t s.t. $\gamma(t) \in U$.

Theorem

(Eriksson-Bique, Soultanis '21) If (X, d, μ) is semi-locally bounded, complete, separable and X has Hausdorff dimension $d_{\mathrm{Haus}} < \infty$, then, there exist p-weak charts $(U_i, \phi_i : X \to \mathbb{R}^{n_i})$ with

1. $\sup_{i\in\mathbb{N}} n_i \leq d_{\mathrm{Haus}}$.
2. $\mu(X \setminus \bigcup U_i) = 0$.

$$f(\gamma_1) - f(\gamma_0) = \int_0^1 \sum_{i\in\mathbb{N}} df_x((\phi_i \circ \gamma)'_t) 1_{U_i}(\gamma_t) dt.$$

$$g_f = |df|.$$

Example computation

Lemma

If $f \in W^{1,p}(X)$ and f is bounded with compact support, then $f^2 \in W^{1,p}(X)$ and

$$d(f^2) = 2fdf$$

Proof: Let (U, ϕ) be a chart

$$(f^2 \circ \gamma)'_t \overset{AC}{=} 2f \cdot (f \circ \gamma)'_t \overset{Chart}{=} 2f(\gamma_t) \cdot df_x((\phi \circ \gamma)'_t).$$

By uniqueness, the claim follows.

Finite Hausdorff dimension?

Definition

p-independent: $U \subset X, \varphi : X \to \mathbb{R}^n$ Lipschitz, s.t. for any countable dense set $V \subset S^{n-1}$, we have for a.e. $x \in U$ that

$$\mathrm{essinf}_{v \in S^{n-1}} g_{v\cdot\varphi} = \inf_{v \in V} g_{v\cdot\varphi}(x) > 0.$$

Theorem

If $(U, \varphi : X \to \mathbb{R}^n)$ is p-independent, then $d_{\mathrm{Haus}}(U) \geq n$.

$\Longrightarrow$ maximal p-independent maps

Finite Hausdorff dimension?

Theorem

If (U, φ) is a maximal p-independent chart, then (U, φ) is a p-weak chart. That is, every $f \in N^{1,p}(X)$ has a unique differential w.r.t. (U, φ).

Consider the map $(U, (\varphi, f))$, which is no longer p-independent.