Nonlinear potential theory, p-harmonic and Green functions on metric spaces

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Classical potentials in $\mathbf{R}^n$, $n \geq 3$, $\nu =$ measure

$$u(x) = U^\nu(x) = \int \frac{d\nu(y)}{|x-y|^{n-2}}$$

harmonic in $\mathbf{R}^n \setminus \operatorname{supp} \nu$: $\quad \Delta u = 0$

- locally minimizes energy

$$\int_G |\nabla u|^2 \, dx \leq \int_G |\nabla v|^2 \, dx \tag{1}$$

$\forall v$ with $v = u$ on ∂G and $\forall$ open $G \Subset \mathbf{R}^n \setminus \operatorname{supp} \nu$

superharmonic in $\mathbf{R}^n$: $\quad -\Delta u = \nu \geq 0 \qquad (-u \text{ subharm})$

- if bdd (or otherwise controlled): (1) holds $\forall v \geq u$ in G with $v = u$ on ∂G and $\forall$ open $G \Subset \mathbf{R}^n$
- lsc and finely cont in $\mathbf{R}^n$: $\{y : |u(y) - u(x)| \geq \varepsilon\}$ is thin at x (in capacitory sense through a Wiener integral)

$\operatorname{cap}(K) = \sup \nu(K)$, taken over all ν with $U^\nu \leq 1$

Nonlinear theory

- p-harmonic functions = solutions of p-Laplace equation

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

and local minimizers of p-energy $\displaystyle\int_\Omega |\nabla u|^p \, dx$.

- Fundamental solution $u(x) = |x-y|^{\frac{p-n}{p-1}}$ for $-\Delta_p u = C_{n,p}\delta_y$ in $\mathbf{R}^n$.

Generalizations:

- Nonhomogeneous materials: $dx \quad \rightsquigarrow \quad w\,dx$ with a weight w
- Manifolds and their Gromov–Hausdorff limits $\rightsquigarrow$ non-smooth spaces
- SubRiemannian geometry, subelliptic equations
- Graphs

Unified approach: Metric space (X, d, μ)

$d =$ metric
$\mu =$ Borel regular measure s.t. $0 < \mu(B) < \infty \quad \forall$ balls $B \subset X$

Heinonen, Koskela, MacManus, Shanmugalingam,1998:

- $g \geq 0$ is a (p-weak) upper gradient of $u : X \to \mathbf{R}$ if

$$|u(x) - u(y)| \leq \int_\gamma g\, ds$$

  for (p-almost) all rectifiable curves γ in X.
  $(x, y =$ endpoints of $\gamma)$
- $\exists$ minimal g_u (in L^p and pointwise a.e.)

$X =$ open set in $\mathbf{R}^n$: $g_u = |\nabla u|$ a.e.

Shanmugalingam, 1998: Sobolev (Newtonian) space

$$N^{1,p}(X) = \left\{ u : \int_X (|u|^p + g_u^p)\, d\mu < \infty \right\}$$

$X = (E, d|_E, \mu|_E)$ gives $N^{1,p}(E)$ for any measurable $E \subset X$

Cheeger 1999: equiv definition for $p > 1$

p-harmonic functions in (open) $\Omega \subset X$:

minimize p-energy, $1 < p < \infty$:

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_{u+\varphi}^p \, d\mu \quad \forall \varphi \in \mathrm{Lip}_c(\Omega)$$

few rectifiable curves in X or ´´bad´´measure $\Rightarrow$
$g_u \equiv 0 \; \forall u$ and hence $N^{1,p}(X) = L^p(X)$

Assumptions for a reasonable theory ($g_u \equiv 0$ $\forall u$ no good):

- μ doubling: $\mu(2B) \le C\mu(B) \quad \forall$ balls $B \subset X$
- p-Poincaré inequality (p-PI): $\forall$ balls $B \subset X$ and $\forall u$

$$\fint_B |u - u_B| \, d\mu \le C \operatorname{diam} B \left(\fint_{\lambda B} g_u^p \, d\mu \right)^{1/p},$$

where $u_B = \fint_B u \, d\mu$

- (X complete) (or local versions)

Cheeger: Possible to define a differentiable structure on X with a vector-valued differential Du and the equation (in weak sense)

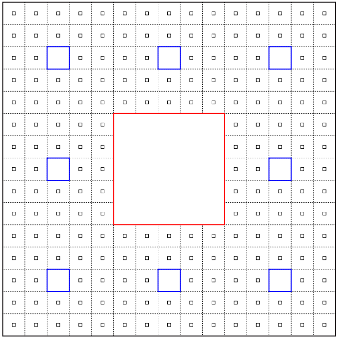
$$-\operatorname{div}(|Du|^{p-2} Du) = 0 \quad \text{or} \quad = \nu$$

Du more abstract than g_u which has a clear geometric meaning.

Examples

- "Nice" open/closed sets in $\mathbf{R}^n$ (with a weight $w\,dx$)
- Manifolds, Heisenberg and Carnot groups
- Laakso spaces
- Hyperbolic fillings
- Sierpiński sponge in $\mathbf{R}^d$ (Ericsson-Bique–Gong, 2021):

(Carpet in $d = 2$:
Mackay–Tyson–Wildrick, 2013)



Scale factors $a_n = \frac{1}{\text{odd number}}$ with

$$\sum_{n=1}^{\infty} a_n^d < \infty$$

Here $a_1 = \frac{1}{3}$, $a_2 = \frac{1}{5}$, $a_3 = \frac{1}{7}$.

dx = Doubling + p-PI $\forall p > 1$

- Which spaces support Poincaré inequality? New such spaces from old?
  - JB 2001: If μ doubling + p-PI and w is A_q weight wrt μ then $w\,d\mu$ supports pq-PI.
  - Lahti 2022: For X nice (complete, μ doubling + 1-PI) $\exists c_* > 0$ s.t. if for quasievery $x \in X$,

$$\liminf_{r\to 0} \frac{\mathrm{cap}_1(A \cap B(x,r), B(x,2r))}{\mathrm{cap}_1(B(x,r), B(x,2r))} < c_*,$$

    then $X \setminus A$ also supports 1-PI.
- Other suitable definitions of gradients and Sobolev spaces? Comparisons? Energy minimizers?
  - e.g. Hajłasz α-gradient

$$|u(x) - u(y)| \le d(x,y)^\alpha (h(x) + h(y)), \qquad \alpha > 0$$

    or different gradients h_j at different scales
  - Korevaar–Schoen spaces

Properties of p-harmonic functions I

Bad news:

- g_u only scalar not vector $\Rightarrow$ no Euler–Lagrange equation
- $g_{u+v} \neq g_u + g_v \Rightarrow$ nonlinear problem also for $p = 2$
- Sheaf property? p-harm in U and $V \quad \Rightarrow \quad$ in $U \cup V$?

Good news (Shanmugalingam + Finland + Linköping, 1998–):

- Hölder continuous C^α
- Maximum and comparison principles:
  $u \leq v$ on $\partial\Omega \quad \Rightarrow \quad u \leq v$ in Ω (Note: no linearity!)
- Harnack inequality: $\max_K u \leq C \min_K u$
- Convergence theorems
- Liouville theorem (under global doubling + p-PI):
  $\nexists$ nonconst bdd p-harm functions on X

Properties of p-harmonic functions II

- Solutions to the Dirichlet problem on (bdd) open $\Omega \subset X$

$$\begin{cases} u \ p\text{-harm} & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

  by various methods (variational, Perron, Wiener)
  and for various bdry data:
  Existence and uniqueness if $f \in N^{1,p}(\overline{\Omega})$ or $f \in C(\partial\Omega)$.
  Resolutivity for general f in the Perron method: $\underline{P}f = \overline{P}f$?
  (even in $\mathbf{R}^n$)
- Invariance under small perturbations of f on the boundary:
  If $\mathrm{cap}_p(\{f_1 \neq f_2\}) = 0$ and f_1 as above then $u_1 = u_2$
  (Remember: nonlinear problem!)

Variational capacity: $$\mathrm{cap}_p(E, \Omega) := \inf_{\varphi} \int_X g_\varphi^p \, d\mu$$

with inf over all $\varphi \in N^{1,p}(X)$ s.t. $\varphi = 1$ on E and $\varphi = 0$ outside Ω.

Cantor type example with Area$(\partial\Omega) > 0$ (B–B–S 2015)

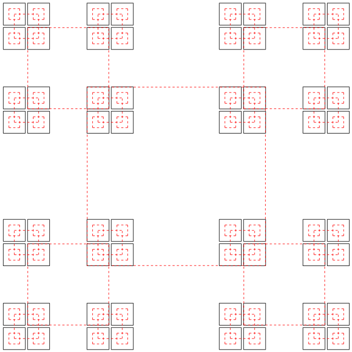
Ω = Square $\setminus$ fat Cantor set

cpt $K = \liminf_{j\to\infty} K_j \subset \partial\Omega$
with full measure in $\partial\Omega$ but

$\overline{\text{cap}}_p^{\Omega}(K) = 0 \quad \forall p \geq 1$
(new capacity – from inside of Ω)

We may perturb boundary data $f \in C(\partial\Omega)$ as we like on K:
Solution of the Dirichlet BVP will not change.

(Ω is regular domain)



Properties of p-harmonic functions III

- Boundary behaviour of p-harm functions and bdry regularity (B–MacManus–S, 2001)
  If $X \setminus \Omega$ not p-thin at $x_0 \in \partial\Omega$ then x_0 is regular for the Dirichlet problem: $\forall f \in C(\partial\Omega)$, the solution u satisfies

$$\lim_{\Omega \ni x \to x_0} u(x) = f(x_0).$$

  Here A is p-thin at x_0 if "Wiener integral"

$$\int_0^1 \left(\frac{\mathrm{cap}_p(A \cap B(x,r), B(x,2r))}{\mathrm{cap}_p(B(x,r), B(x,2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty.$$

- For solutions of $\mathrm{div}(|Du|^{p-2} Du) = 0$ also converse:
  $X \setminus \Omega$ p-thin $\Rightarrow$ x_0 not regular
- Regular points characterized by barriers (B–B, 2006)

p-harmonic functions on bad (measurable) sets

Recall: $(E, d|_E, \mu|_E)$ gives $N^{1,p}(E)$ for any measurable $E \subset X$s

- Dirichlet problem for minimizing p-energy $\int_E g_u^p \, d\mu$ with bdry data $f \in N^{1,p}$ solvable and nontrivial iff fine-int $E \neq \varnothing$.
- In that case, it coincides with the solution for fine-int E.
- (G p-finely open iff $X \setminus G$ is p-thin at every $x \in G$)

Fine potential theory and finely p-harmonic functions on finely open sets (B–B–Latvala (+J. Malý))

- Newtonian functions are finely cont q.e. and quasicont
- Convergence theorems
- Perron method and resolutivity on finely open sets
- Fine continuity for solutions of Dirichlet problem with bdry data $f \in C(X)$.

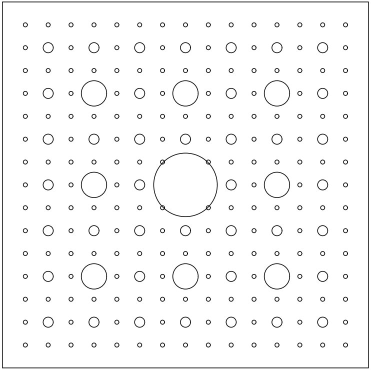
  For general finely p-harm functions (even in $\mathbf{R}^n$)?

Swiss cheese example - nowhere dense p-finely open set

From $[0,1]^n \subset \mathbf{R}^n$ remove $2^{(k-1)n}$ closed balls: $k = 1, 2, \ldots$

- radii $r_k = 2^{-\alpha k}\varepsilon$
- $0 < \varepsilon < \frac{1}{2}$
- $\alpha > n/(n-p)$
- $1 < p < n$

- G p-finely open
- $\operatorname{int} G = \varnothing$
- $|G| > 0$



p-superharmonic functions and some properties

- Main ingredient in Perron method for solving Dirichlet BVP
- As barriers in bdry regularity
- Defined by comparison principle on every $G \Subset \Omega$ with $\operatorname{cap}_p(X \setminus G) > 0$ (when u lsc and $u \not\equiv \infty$ in any component): If $v \in C(\overline{G})$ p-harm in G and $v \le u$ on ∂G then $v \le u$ in G.
- Or equivalently: $\forall k \;\; u_k := \min\{u, k\}$ is lsc-regularized superminimizer of p-energy:

$$\int_\Omega g_{u_k}^p \, d\mu \le \int_\Omega g_{u_k+\varphi}^p \, d\mu \qquad \forall \; 0 \le \varphi \in \operatorname{Lip}_c(\Omega)$$

- Finely continuous (JB, Korte 2008), i.e. sub- and superlevel sets are finely open
- Finely open sets give the coarsest topology making p-superharm functions continuous.
- With Du instead of g_u: p-superharm functions satisfy $-\operatorname{div}(|Du|^{p-2} Du) = \nu$ with Radon measure ν

Singular and Green functions – fundamental solutions

Assume that Ω is bdd and $\mathrm{cap}_p(X \setminus \Omega) > 0$

Definition: Singular function in Ω with singularity at $x_0 \in \Omega$:

- $u > 0$ is p-harm in $\Omega \setminus \{x_0\}$
- u is p-superharm in Ω
- $u = 0$ on $\partial\Omega$ (in the sense of Sobolev spaces)

Green function = properly normalized singular function.

Theorem

- There exists a singular function in Ω with singularity at x_0.
- If u and v are singular functions in Ω with singularity at x_0 then $u \simeq v$ in Ω. Moreover, near x_0,

$$u(x) \simeq \mathrm{cap}_p(B(x_0, r), \Omega)^{1/(1-p)}, \quad \text{where } r = d(x, x_0).$$

Theorem + Definition

For every singular function u there is unique $\alpha > 0$ such that for $\bar{u} = \alpha u$ and all $0 \le a < b \le \bar{u}(x_0)$,

$$\mathrm{cap}_p(\{x : \bar{u}(x) \ge b\}, \{x : \bar{u}(x) > a\}) = (b-a)^{1-p},$$

i.e. $\bar{u}$ is a Green function.

Sharp estimates for cap_p and p-harm functions (B–B–Lehrbäck)

- For all $0 < 2r \le R \le \frac{1}{4} \operatorname{diam} X$, writing $B_r := B(x_0, r)$,

$$\mathrm{cap}_p(B_r, B_R)^{1/(1-p)} \simeq \int_r^R \left(\frac{\rho}{\mu(B_\rho)}\right)^{1/(p-1)} d\rho.$$

- If u is p-harm in $\Omega \setminus \{x_0\}$ and $\lim_{x \to x_0} u(x) = \infty$, then there is $R > 0$ such that near x_0,

$$u(x) \simeq \inf_{B_R} u + \int_{d(x,x_0)}^R \left(\frac{\rho}{\mu(B_\rho)}\right)^{1/(p-1)} d\rho.$$

Exponent sets (dimensions) for μ at x_0:

$$\overline{S}_0 = \{s > 0 : \mu(B_r) \gtrsim r^s \text{ for } 0 < r \le 1\} \quad \text{and } \bar{s}_0 = \inf \overline{S}_0$$

$$\overline{Q}_0 = \left\{ s > 0 : \frac{\mu(B_r)}{\mu(B_R)} \gtrsim \left(\frac{r}{R}\right)^s \text{ for } 0 < r < R \le 1 \right\}.$$

$\underline{S}_0$ and $\underline{Q}_0$ similar but with $\lesssim$ instead of $\gtrsim$

Lebesgue measure in $\mathbf{R}^n$: $\overline{S}_0 = \overline{Q}_0 = [n, \infty)$ and $\underline{S}_0 = \underline{Q}_0 = (0, n]$
In general not equal and can be open.

- $\operatorname{cap}_p(B_r, B_R) \simeq \begin{cases} R^{-p}\mu(B_R), & p > \inf \overline{Q}_0, \\ r^{-p}\mu(B_r), & p < \sup \underline{Q}_0, \end{cases}$
- $C_p(\{x_0\}) \begin{cases} = 0, & p < \bar{s}_0 \text{ or } p = \bar{s}_0 \notin \overline{S}_0 \setminus \underline{S}_0, \\ > 0, & p > \bar{s}_0. \end{cases}$

Integrability for Green and p-harm functions

Let $\quad \bar{s}_0 = \inf \bar{S}_0, \quad \tau_p = \dfrac{\bar{s}_0(p-1)}{\bar{s}_0 - p} \quad$ and $\quad t_p = \dfrac{\bar{s}_0(p-1)}{\bar{s}_0 - 1}$.

Theorem

Assume that $C_p(\{x_0\}) = 0$. Let $u =$ Green function in Ω with singularity at x_0. Then for $B = B(x_0, r) \Subset \Omega$:

- $p \leq \bar{s}_0$ and u is unbdd;
- $u \in L^\tau(B)$ and $g_u \in L^t(B)$ for all $\tau < \tau_p$ and $t < t_p$;
- $u \notin L^\tau(B)$ if $\tau > \tau_p$;
- $g_u \notin L^t(B)$ if $t > t_p$ and μ supports a t-PI.
- if $p = \bar{s}_0$, then $g_u \in L^t(B)$ iff $0 < t < p$;

Same (non)integrability conclusions hold if $u \geq 0$ is a general p-harm function in $\Omega \setminus \{x_0\} \subset X$ with $\lim_{x \to x_0} u(x) = \infty$.

Thank you!