Exploring Temporal Pulse Replication in the Fitzhugh-Nagumo Equation

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Introduction

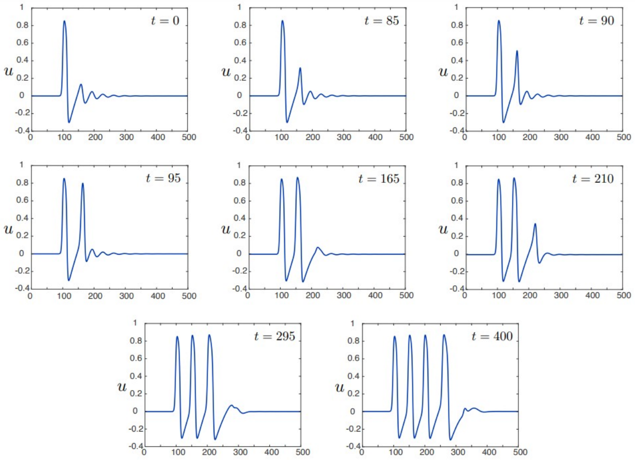

Figure: Carter *et al.* 2021

Overview

1. Constructing the Banana
2. Stability of Pulses
3. Analysis

The Fitzhugh-Nagumo Equation

$$
\begin{aligned}
u_t &= u_{xx} + u(u-a)(1-u) - w \\
w_t &= \epsilon(u - \gamma w)
\end{aligned}
$$

Traveling wave solutions depend only on $\zeta = x + ct$, yielding

$$
\begin{aligned}
u_\zeta &= v \\
v_\zeta &= cv - u(u-a)(1-u) + w \\
w_\zeta &= \frac{\epsilon}{c}(u - \gamma w)
\end{aligned}
$$

Slow Drift: Critical Manifold

In our problem, we note that the critical manifold is given by $v = 0$, $w = u(u - a)(1 - u)$. The manifold can be divided into a left, center, and right part by two fold points located at the extrema of the cubic.

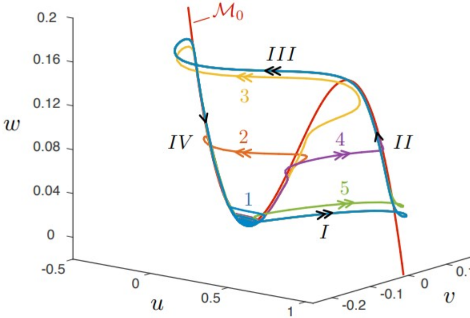

Figure: Carter *et al.* 2021

Normal Stability of Branches

$$f = \begin{pmatrix} v \\ cv - u(u-a)(1-u) + w \end{pmatrix}$$

$$Df = \begin{pmatrix} 0 & 1 \\ -\partial_u\big(u(u-a)(1-u)\big) & c \end{pmatrix}$$

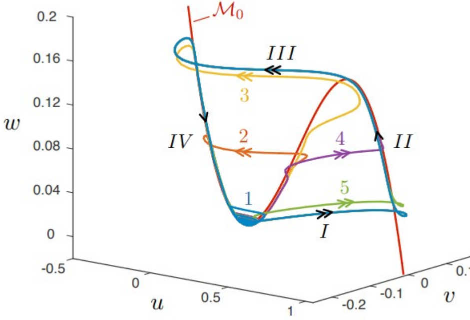

Figure: Carter *et al.* 2021

Fast Jumps: The Nagumo Front and Back

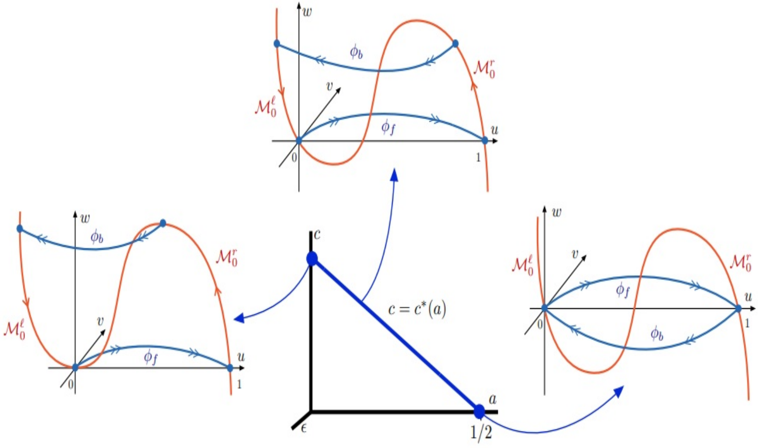

Figure 6: *Shown are the singular fronts* ϕ_f *and* ϕ_b *for the layer problem* (2.7) *for* $\epsilon = 0$ *and* $0 \leq a \leq 1/2$.

Figure: Carter *et al.* 2018

Summary of Result

Theorem (Carter *et al*, 2018)

For each $0 < \gamma < 4$ and each sufficiently small $\epsilon > 0$, there exists a one-parameter family of traveling pulses (parametrized by $s \in [0, 8/27]$) which is C^1 in $(s, \sqrt{\epsilon})$. For s sufficiently small, the solutions are one-pulses with oscillatory tails while for s sufficiently close to $8/27$ they are double pulses. Away from either endpoint, a and c satisfy

$$(a, c)(s, \epsilon) = (a_*, c_*)(\epsilon) + O(e^{-q/\epsilon})$$

for appropriately chosen a_*, c_*.

Visualizing the Transition

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Spectral Stability of Pulses

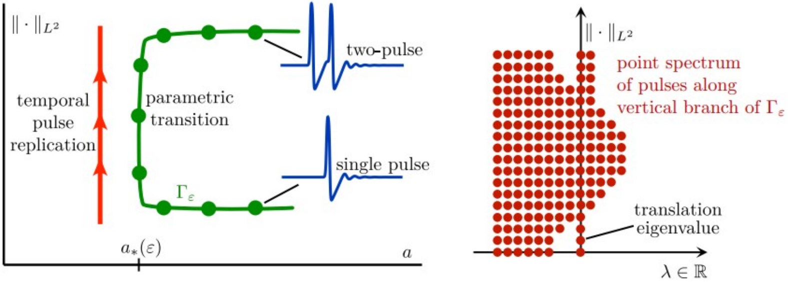
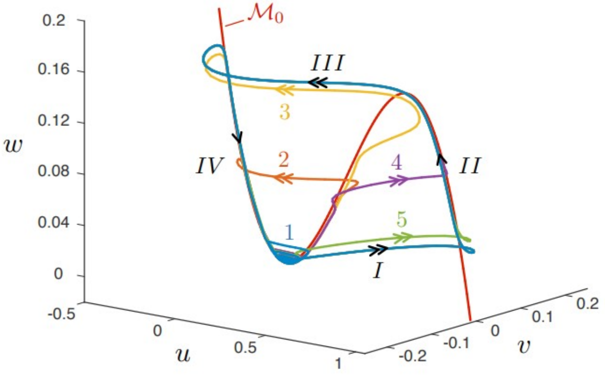


Figure: Carter *et al.* 2021

Intuition for Spectral Analysis

1. Solutions jump left or right at height s
2. The speed at which solutions travel along the slow manifold is $O(\frac{1}{\epsilon})$
3. Points 1 and 2 imply that eigenfunctions with support along the middle branch solve a boundary value problem with domain of size $O(\frac{1}{\epsilon})$
4. The spectra of an operator on a bounded domain of size $O(\frac{1}{\epsilon})$ as $\epsilon \to 0$ accumulates on the absolute spectrum of the operator
5. The absolute spectrum from the center part of the critical manifold reaches into the right-half plane

Jump Height and s



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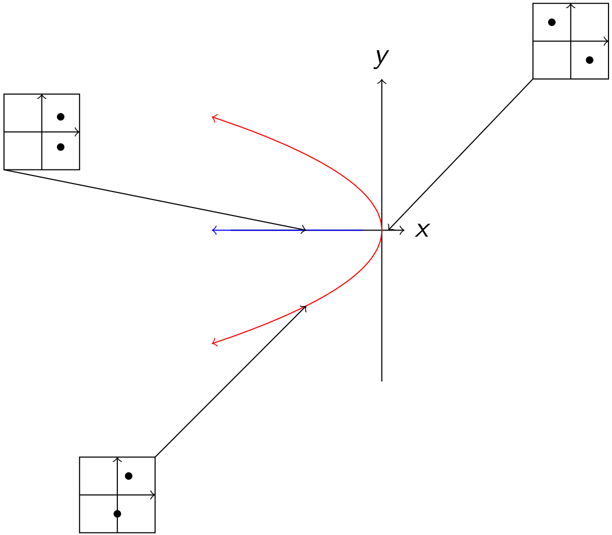
Absolute Spectrum



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Questions to Answer

Suppose we consider solutions to the Fitzhugh-Nagumo equations of the form $y(x,t) = \Gamma_\epsilon(s(t))(x) + v(x,t)$, where $y = (u,w)$.

1. Can we determine the speed of travel $\frac{ds}{dt}$ along the banana?
2. Can we prove that v is small in some appropriate norm?

Taylor Expansion

We formally write the Fitzhugh-Nagumo equations as an ODE on an appropriate Banach space

$$\frac{dy}{dt} = f(y, \mu)$$

where $\mu = (a, c)$. We denote the values of the parameters along the banana itself by μ^*. Expanding, we find that

$$\frac{ds}{dt} \cdot \Gamma_{\epsilon}^{'} + \frac{dv}{dt} = D_y f(\Gamma_{\epsilon}, \mu^*)v + D_{\mu} f(\Gamma_{\epsilon}, \mu^*)(\mu - \mu^*) + \text{h.o.t.}$$

Finally, projecting in the direction of a vector p yields

$$\frac{ds}{dt}\langle\Gamma_{\epsilon}^{'}, p\rangle + \langle\frac{dv}{dt}, p\rangle = \langle D_y f(\Gamma_{\epsilon}, \mu^*)v, p\rangle + \langle D_{\mu} f(\Gamma_{\epsilon}, \mu^*)(\mu - \mu^*), p\rangle + \text{h.o.t.}$$

Bibliography

[1] P Carter and B Sandstede, "Unpeeling a homoclinic banana", *SIAM Journal on Applied Dynamical Systems*, 2018

[2] P Carter, J Rademacher, B Sandstede, "Pulse replication and accumulation of eigenvalues", *SIAM Journal on Mathematical Analysis*, 2021

[3] B Sandstede and A Scheel, "Absolute and convective instabilities of waves on unbounded and large bounded domains", *Physica D*, 2000