Frobenius objects in Rel and Span

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What is a Frobenius object?

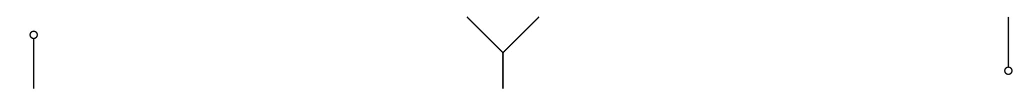
Let $\mathcal{C}$ be a monoidal category with monoidal unit 1 and monoidal product $\otimes$. A **Frobenius object** in $\mathcal{C}$ is an object $X \in \mathrm{Ob}(\mathcal{C})$ equipped with morphisms

- $\eta : 1 \to X$ (Unit)
- $\mu : X \otimes X \to X$ (Multiplication)
- $\varepsilon : X \to 1$ (Counit)

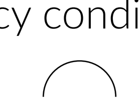
satisfying the following axioms:

- **Unitality:** $\mu \circ (\mathbf{1} \times \eta) = \mu \circ (\eta \times \mathbf{1}) = \mathbf{1}$
- **Associativity:** $\mu \circ (\mathbf{1} \times \mu) = \mu \circ (\mu \times \mathbf{1})$
- **Nondegeneracy:** There exists $\beta : 1 \to X \otimes X$ such that $(\varepsilon \otimes \mathbf{1}) \circ (\mu \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \beta) = (\mathbf{1} \otimes \varepsilon) \circ (\mathbf{1} \otimes \mu) \circ (\beta \otimes \mathbf{1}) = \mathbf{1}$.

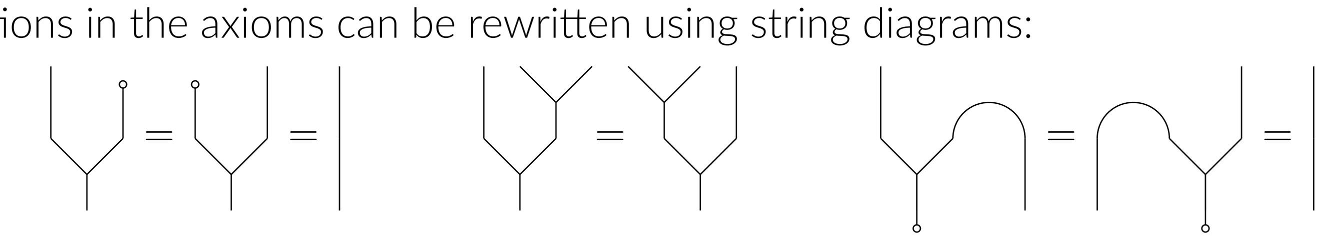
It can be helpful to use string diagrams to describe morphisms built out of the above. The unit, multiplication, and counit are denoted by the following diagrams (read from top to bottom):



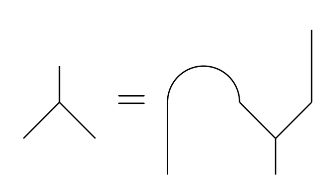
It can be proven that β in the nondegeneracy condition is unique. It is denoted by



The equations in the axioms can be rewritten using string diagrams:



Comultiplication can be defined by



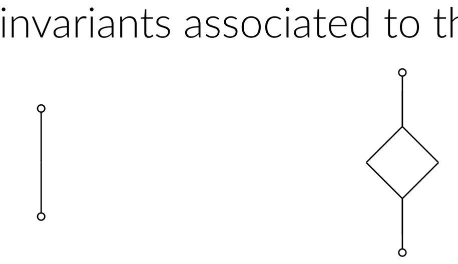
and you can use the above axioms to prove that it is counital and coassociative.

If $\mathcal{C}$ is symmetric monoidal, then we can define **commutative** Frobenius objects.

Why Frobenius objects?

A well-known result (see [5]) is that commutative Frobenius objects in $\mathcal{C}$ correspond to $\mathcal{C}$-valued 2-dimensional topological field theories (i.e. symmetric monoidal functors from the 2D cobordism category to $\mathcal{C}$.)

In particular, a commutative Frobenius object gives invariants of closed surfaces. The invariants are in $\mathrm{Mor}_{\mathcal{C}}(1,1)$. For example, the invariants associated to the sphere and torus are

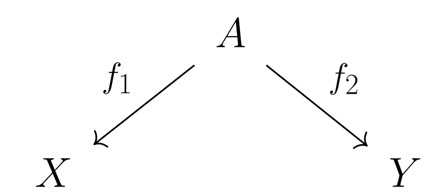


Noncommutative Frobenius objects appear in related situations, e.g. in extended TFTs.

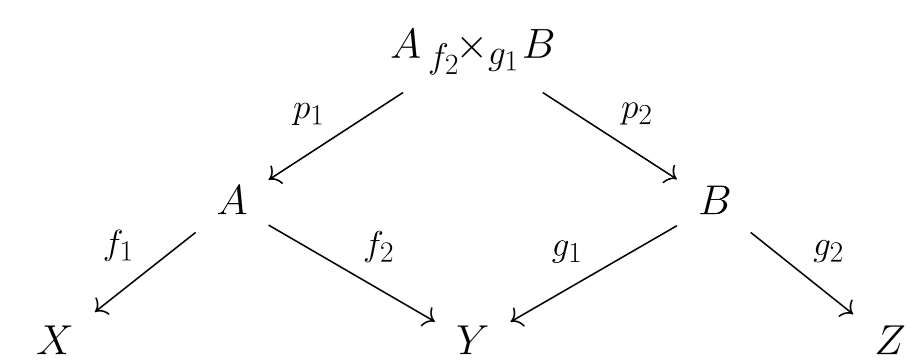
This is based on joint work with Ruoqi Zhang [7], with Ivan Contreras and Molly Keller [2], and works in progress with Ivan Contreras, Adele Long, Sophia Marx, and Walker Stern.

What are Rel and Span?

- **Rel** is the category whose objects are sets, and where a morphism from X to Y is a relation $R \subseteq X \times Y$. If $S \subseteq Y \times Z$ is a morphism from Y to Z, then the composition $S \circ R \subseteq X \times Z$ is given by
$$S \circ R = \{(x,z) : (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in Y\}.$$
- **Span** is the category whose objects are sets, and where a morphism from X to Y is an isomorphism class of spans



Composition is given by pullback:



The Cartesian product gives both **Rel** and **Span** the structure of a symmetric monoidal category.

There is a natural symmetric monoidal functor **Span** $\to$ **Rel**, so a Frobenius object in **Span** induces a Frobenius object in **Rel** (but the converse isn't true).

Why Rel and Span?

Our main motivation is that **Rel** and **Span** can be seen as set-theoretic models for the **symplectic category**.

In principle, the the symplectic category has symplectic manifolds as objects and Lagrangian relations as morphisms. But this definition isn't well-defined because Lagrangian relations don't compose nicely.

- If you simply ignore the geometry, then you get **Rel**.
- A rigorous definition of the symplectic category was given by Wehrheim and Woodward [9]. The morphisms are more sophisticated (formal compositions of Lagrangian relations modulo an equivalence relation). Li-Bland and Weinstein [6] gave a general formulation of the Wehrheim-Woodward construction. When you apply this construction to **Rel**, you get **Span**.

Although **Rel** and **Span** seem simple, you can get nontrivial invariants from them! In both cases, the monoidal unit is the one-point set $\{\bullet\}$.

- There are exactly two relations from $\{\bullet\}$ to itself: the empty one and the one that's nonempty. They can be viewed as booleans ("false" and "true", respectively). So the invariants coming from a commutative Frobenius object in **Rel** are booleans.
- A span from $\{\bullet\}$ to $\{\bullet\}$ is given by set (up to isomorphism). So the invariants coming from a commutative Frobenius object in **Span** are cardinalities. Note: in the special case where the spans are finite, then the invariants are natural numbers.

Main result

Frobenius objects in **Rel** and **Span** can be encoded as simplicial sets equipped with an automorphism $\hat{\alpha}$ of the set of 1-simplices. The simplicial sets that arise from Frobenius objects satisfy conditions that are similar to (but weaker than) the 2-Segal conditions (c.f. [3, 4, 1, 8]), with an additional compatibility condition for $\hat{\alpha}$.

Important example

An important class of examples comes from **groupoids**. The nerve of a groupoid satisfies the conditions for a Frobenius object in **Span**. One can take $\hat{\alpha}$ to be the inverse map, but more generally one can use any section of the target map to "twist" the inverse map.

In fact: A symplectic groupoid can be seen as a Frobenius object in the symplectic category!

More examples

There are examples of Frobenius objects in **Rel** and **Span** that don't come from groupoids. There are constructions coming from compact oriented Riemannian manifolds, from conjugacy classes of groups, and more. Additionally, we have classified Frobenius objects in **Rel** with 2 or 3 elements. There are 5 Frobenius objects in **Rel** with 2 elements, and 23 Frobenius objects in **Rel** with 3 elements.

Calculations of invariants

Invariants arising from commutative Frobenius objects in **Rel** and **Span** can often be explicitly calculated. Let $Z(\Sigma_g)$ denote the invariant associated to the genus g surface.

- Let G be a finite abelian group with $\hat{\alpha}(g) = g^{-1}\omega$ for any fixed $\omega \in G$. If we think of G as a Frobenius object in **Span**, then
$$Z(\Sigma_g) = \begin{cases} |G|^g & \text{if } \omega^g = \omega, \\ 0 & \text{otherwise.} \end{cases}$$
- There is an infinite family of 2-element Frobenius objects in **Span**, parametrized by $n \in \mathbb{N}$, where
$$Z(\Sigma_g) = \frac{(n^2+2)^g + n^2}{1+n^2}.$$

References

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