Riemannian metrics and properness for Lie groupoids

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> If a groupoid $\mathcal{G} \rightrightarrows M$ admits a Riemannian metric compatible with its structure, how far is it to be proper?

A possibility is look for "bi-invariant metrics" following the Lie group case. But, since such compatibility remains a unclear condition our approach is based on see the groupoid as symmetries of the metrics.

Examples

1) Isometric actions

Myers-Steenrood

(M,η) Riemannian manifold $\Rightarrow$ $Iso(M,\eta)\ltimes M \rightrightarrows M$

$Iso(M,\eta)$ isometry group — proper Lie groupoid

$G \curvearrowright (M,\eta)$ by isometries

$G \xrightarrow{\varphi} Iso(M,\eta)$ $\Rightarrow$

$\boxed{\begin{array}{l} \overline{Im\,\varphi}\ltimes M \rightrightarrows M \\ \text{proper Lie groupoid} \\ G\ltimes M \xrightarrow{\Phi} \overline{Im\,\varphi}\ltimes M \\ (g,x) \mapsto (\varphi(g),x) \\ \text{subgroupoid with } \Phi(G\ltimes M) \text{ dense} \end{array}}$

2) Riemannian foliations

$(M,\mathcal{F},\eta)$

Bott connection

$Hol\,\mathcal{F} \curvearrowright \nu\mathcal{F}$

$Hol\,\mathcal{F} \xrightarrow{\Phi} O(\nu\mathcal{F})$

Molino $\Rightarrow$

$\boxed{\begin{array}{l} \overline{Im\,\Phi} \rightrightarrows M \quad \text{proper Lie groupoid} \\ Hol\,\mathcal{F} \xrightarrow{\Phi} \overline{Im\,\Phi} \\ \text{subgroupoid with } \Phi(Hol\,\mathcal{F}) \text{ dense} \end{array}}$

$O(\nu\mathcal{F}) = \{\nu_x\mathcal{F} \xrightarrow{g} \nu_y\mathcal{F} : \text{linear isometry}\}$

$\downdownarrows$

M

Results

$\mathcal{V}_x O \xrightarrow{g} \mathcal{V}_y O$ isometry for all g in $\mathcal{G}$

$\mathcal{G} \rightrightarrows M$ Lie groupoid and η an invariant metric on M
(0-metric)

If η is complete and $O \subset M$ orbit $\xRightarrow{\text{Molino}}$ $\begin{cases} B = \overline{O} & \text{closed invariant submanifold} \\ O' \subset B \text{ orbit} & \text{then } B = \overline{O'} \\ \mathcal{G}_B \rightrightarrows B & \text{regular Lie groupoid} \end{cases}$

Theorem (-, Struchiner) Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with a complete invariant metric η. If $\mathcal{G}$ admits an invariant linearization around $B = \overline{O}$, then there is a $\overline{\mathcal{G}} \rightrightarrows U$ proper Lie groupoid over an invariant neighbourhood U of B such that $\mathcal{G}_U$ is a dense Lie subgroupoid of $\overline{\mathcal{G}}$, i.e., there exists $\mathcal{G}_U \xrightarrow{\Phi} \overline{\mathcal{G}}$ with $\overline{\Phi(\mathcal{G}_U)} = \overline{\mathcal{G}}$.

Obs: Φ is not necessarily injective!

Results

Corollary: If $\mathcal{G} \rightrightarrows M$ admits a complete 2-metric, then there is a $\bar{\mathcal{G}} \rightrightarrows U$ proper Lie groupoid over an invariant neighbourhood U of B such that $\mathcal{G}_U$ is a dense subgroupoid of $\bar{\mathcal{G}}$.

Theorem sketch proof:

Step 1: $\mathcal{G}_U \cong \mathcal{G}_B \ltimes \nu B$, we show that $\mathcal{G}_B \xrightarrow{\Phi} O(\nu B)$ is a Lie subgroupoid

step 2: $\mathcal{G}_B \Rightarrow \mathcal{F}$ right invariant Riemannian foliation on $O(\nu B)$

step 3: by Molino $\bar{\mathcal{F}}$ is given by the orbits of the proper Lie groupoid $\overline{\mathrm{Hol}\,\mathcal{F}}$.

step 4: quotient by $O(\nu B)$:

$$\begin{tikzcd} \mathcal{G}_B \arrow[r] \arrow[rr, bend right, "\Phi"'] & \mathrm{Hol}\,\mathcal{F}/O(\nu B) \arrow[r] & \overline{\mathrm{Hol}\,\mathcal{F}}/O(\nu B) \end{tikzcd}$$

Further directions

$\mathcal{G} \rightrightarrows M$ Lie grouoid with an invariant metric

$$\mathcal{O}(\mathcal{G}) := \{ \nu_x \mathcal{O} \xrightarrow{f} \nu_{x'} \mathcal{O}' : f \text{ linear isometry} \} \rightrightarrows M$$ (not Lie groupoid!)

normal representation $\Rightarrow$ $\mathcal{G} \to \mathcal{O}(\mathcal{G})$ groupoid map

Use our local result to construct a proper Lie groupoid $\tilde{\mathcal{G}}$ fitting in a commutative triangle

$$\begin{tikzcd}
\mathcal{G} \arrow[rr] \arrow[dr, "\Phi"'] & & \mathcal{O}(\mathcal{G}) \\
& \mathcal{G} \arrow[ur] &
\end{tikzcd}$$

Where Φ is a dense subgroupoid (not necessarily injective).