Equivariant cohomology for 2-group actions and stacky group actions

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Abstract

We consider a strict action of a Lie 2-group on a Lie groupoid and we look for a notion of equivariant cohomology. Consequently, we check this cohomology is Morita invariant, giving us the relation with the category of differentiable stacks.

Introduction

The idea of equivariant cohomology is important when looking for a notion of cohomology for the quotient space

$$M/G$$

when G is a Lie group acting on a smooth manifold M. We know that this quotient is not always a smooth manifold and the cohomology might not be well-defined. However, there exist different models that allow us to compute the cohomology of these spaces. For instance, the Borel model provides an option considering the use of the space

$$EG \times_G M$$

and the property of contractible space of EG to get the cohomology of M/G.
In order to find the equivariant cohomology for 2-group actions, we consider a Lie 2-group

$$\mathbb{G} = (G_1 \rightrightarrows G_0)$$

acting on Lie groupoid $\mathbb{X} = (X_1 \rightrightarrows X_0)$ and we consider its transformation double Lie groupoid. Then we can use the nerve of this double lie groupoid and with this bisimplicial smooth manifold, we compute the cohomology using differential forms on this bisimplicial smooth manifold. After this we go to the category of differentiable stacks and we study the notion of stacky Lie group, actions of stacky Lie groups on differentiable stacks and the implications of these actions in this category.

We state that

A first approach

1. Let $\mathbb{G}$ be a Lie 2-group acting on a Lie groupoid $\mathbb{X}$.
2. Consider the transformations double Lie groupoid given by

$$\begin{array}{ccc} G_1 \times X_1 & \rightrightarrows & X_1 \\ \downdownarrows & & \downdownarrows \\ G_0 \times X_0 & \rightrightarrows & X_0 \end{array} \qquad (1)$$

   where the vertical Lie groupoids is given by the cartesian product groupoid of $\mathbb{G}$ and $\mathbb{X}$, and the horizontal Lie groupoids are the transformation groupoids given by the actions of G_1 and G_0 on X_1 and X_0, respectively.
3. Then we can consider the nerve of this double Lie groupoid that is,

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ X_2 & \Leftarrow & G_2 \times X_2 & \Lleftarrow & G_2^2 \times X_2 & \cdots & \dots \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ X_1 & \Leftarrow & G_1 \times X_1 & \Lleftarrow & G_1^2 \times X_1 & \cdots & \dots \\ \downdownarrows & & \downdownarrows & & \downdownarrows & & \\ X_0 & \Leftarrow & G_0 \times X_0 & \Lleftarrow & G_0^2 \times X_0 & \cdots & \dots \end{array}$$

With this we get triple complex $C^{\bullet,\bullet,\bullet}$ with $C^{p,q,n} = \Omega^p(G_n^q \times X_n)$ where the differentials are given by the exterior derivative, d, and the two differentials that come from the bisimplicial structure, ∂_q and ∂_n.

A cohomology for 2-group actions

1. If we consider the total complex $C^\bullet$, where $C^m = \bigoplus_{m=p+q+n} C^{p,q,n}$ with differential $D = d + (-1)^p\partial_q + (-1)^{p+q}\partial_n$, its cohomology will be denoted by $H(\mathbb{G} \circlearrowright \mathbb{X})$.
2. Let $\mathbb{G}$ be a 2-group acting on $\mathbb{X}$ and $\mathbb{Y}$. A $\mathbb{G}$-equivariant groupoid morphism $\phi : \mathbb{X} \to \mathbb{Y}$ induces a map $H(\mathbb{G} \circlearrowright \mathbb{X}) \to H(\mathbb{G} \circlearrowright \mathbb{Y})$ which is an isomorphism if ϕ is Morita.

Equivariant cohomology for 2-group actions

So we define the **equivariant cohomology for the 2-group action of $\mathbb{G}$ on $\mathbb{X}$** by

$$H_{\mathbb{G}}(\mathbb{X}) := H(\mathbb{G} \circlearrowright \mathbb{X}).$$

What does it happen when we work in the category of Differentiable stacks? For this we need to consider the notion of stacky Lie group and stacky Lie group actions as introduce by Bursztyn-Noseda-Zhu, [2],

Stacky Lie groups

A **stacky Lie group** is a stacky Lie groupoid over a point. That is, a differentiable stack $\mathcal{G}$ with the following structure:

$$\mathcal{G} \rightrightarrows * \xrightarrow{1} \mathcal{G} \xrightarrow{i} \mathcal{G}, \quad i(g) = g^{-1}$$

$$\mathcal{G} \times \mathcal{G} \xrightarrow{M} \mathcal{G}, \quad M(g,h) = gh.$$

with 2-isomorphisms $\alpha, \lambda, \rho, 1_l, 1_r$ that represents weaker forms of associativity, identity and inverse axioms. These 2-isomorphisms accomplish some higher coherence conditions for the associativity, identity and inverse axioms.

Homomorphism of stacky Lie groups

A **homomorphism of stacky Lie groups** is a morphism of differentiable stacks $(\phi, \varepsilon, \epsilon, E) : \mathcal{G}_1 \to \mathcal{G}_2$ which preserves the structure of stacky Lie groups, where ε, ϵ, E are 2-isomorphisms for the associativity, identity and inverse axioms accomplishing some higher coherence conditions.

Example

If $\mathbb{G}$ Lie 2-group, then the canonical atlas

$$G_0 \xrightarrow{q} B\mathbb{G} \qquad (2)$$

where $B\mathbb{G}$ is its classifying stack is a homomorphism of stacky Lie groups with 2-isomorphisms given by identities.

We recall the notion of stacky Lie group actions,

Actions of stacky Lie groups

An **action of a stacky Lie group** $\mathcal{G}$ on a differentiable stack $\mathfrak{X}$ is a morphism of stacks:

$$A : \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$$

with two 2-isomorphisms β and ξ that encode the two ways that two elements of $\mathcal{G}$ act on $\mathfrak{X}$ and the action of the identity accomplishing some higher coherence conditions. We say that $\mathfrak{X}$ is $\mathcal{G}$-stack.

In a similar way it is possible to have a notion of **equivariant morphism** (F, δ) **with respect to a homomorphism** $(\phi, \varepsilon, \epsilon, E)$. With this we get the following

Proposition

Let $\mathbb{G}$ be a 2-group acting on $\mathbb{X}$. We have that the canonical atlas $X_0 \xrightarrow{p} BX$ is an equivariant morphism with respect to the homomorphism $G_0 \xrightarrow{q} B\mathbb{G}$ as in (2).

What is happening when $B\mathbb{G}$, with $\mathbb{G}$ a Lie 2-group, is acting on a differentiable stack $\mathfrak{X}$? We provide a definition

BG-atlas

The $B\mathbb{G}$-stack, $\mathfrak{X}$, is a differentiable $B\mathbb{G}$-stack if there exists a smooth manifold X_0 with a smooth action $a_0 : G_0 \times X_0 \to X_0$ and a equivariant morphism $(p, \sigma) : X_0 \to \mathfrak{X}$ between the G_0-stack, X_0, and the $B\mathbb{G}$-stack, $\mathfrak{X}$, with respect to the homomorphism of stacky Lie groups, $q : G_0 \to B\mathbb{G}$.

Proposition

Let $\mathfrak{X}$ be a differentiable $B\mathbb{G}$-stack with $B\mathbb{G}$-atlas given by $(p, \sigma) : X \to \mathfrak{X}$. Then there exists a strict $(G_0 \times_{B\mathbb{G}} G_0)$-action on $X_0 \times_{\mathfrak{X}} X_0$.

If we consider the associated Lie groupoids $\mathbb{G}$ and $\mathbb{X}$ of the stacks $B\mathbb{G}$ and $\mathfrak{X}$ respectively, then we have:

Proposition

$\mathbb{G}$ is a Lie 2-group acting on the Lie groupoid $\mathbb{X}$.

Definition

If $\mathfrak{X}$ is a differentiable $B\mathbb{G}$-stack with $B\mathbb{G}$-atlas given by $(p, \sigma) : X \to \mathfrak{X}$, the **equivariant cohomology** of $\mathfrak{X}$ with respect to $B\mathbb{G}$ is defined by:

$$H^*_{B\mathbb{G}}(\mathfrak{X}) = H_{\mathbb{G}}(\mathbb{X}).$$

This extends the notion of equivariant cohomology for a Lie group G acting on a differentiable stack given in [1].

In the search of another model for this cohomology, we consider a Lie groupoid $\mathbb{X}$ with a Lie 2-group action by $\mathbb{G}$, where G_1 is a compact connected Lie group.

Cartan Model

1. We consider the double complex $(C^{\bullet,\bullet}, d_{G_n}, \partial)$ given by:

$$C^{n,k} = \Omega^k_{G_n}(X_n) = \left(\bigoplus_{k=2p+q} (S^p(\mathfrak{g}_n^\vee) \otimes \Omega^q_{dR}(X_n))^{G_n} \right),$$

   where $\mathfrak{g}_n^\vee$ denotes the dual of the Lie algebra of G_n, d_{G_n} is the Cartan differential and ∂ is the operator given by simplicial structure.
2. Using a spectral sequence argument we can see that this cohomology computes the same cohomology as $H_{\mathbb{G}}(\mathbb{X})$.

References

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