Specialisations of families of rational maps

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Introduction

In this talk we are concerned with rational maps.

Rational maps

$$f : \mathbb{P}_1 \to \mathbb{P}_1$$

$$f = \frac{a_0 z^{d_1} + \cdots a_{d_1}}{b_0 z^{d_1} + \cdots + a_{d_2}} \in \overline{\mathbb{Q}}(z), \quad \deg(f) = \max\{d_1, d_2\}.$$

$f^{\circ n} = f(f^{\circ n-1}), f^{\circ 0} = \mathrm{Id}.$

Warning: The symbol z sometimes denotes a variable and sometimes a closed point. We will freely pass from endomorphisms to rational functions and back. We also often assume that the field under consderations is embedded into $\mathbb{C}$.

Examples:

$$f_1 = z^2 - 2, f_2 = z^2, f_3 = z^2 - 1.$$

The maps f_1, f_2 are *exceptional*. There exist dominant maps

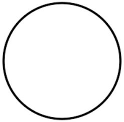
$$\pi_i : \mathbb{G}_m \to \mathbb{P}_1, i = 1, 2$$

such that $\pi_i \circ [2] = f_i \circ \pi_i, i = 1, 2$, where $[2] : \mathbb{G}_m \to \mathbb{G}_m$ is multiplication by 2 on the algebraic group $\mathbb{G}_m$. For the map f_3 no such maps exist.

Julia set

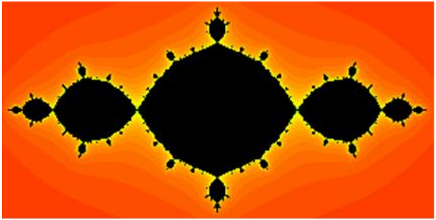
$$J(f) = \partial\{z \in \mathbb{C}; |f^{\circ n}(z)| \nrightarrow \infty\}$$

Julia set of f_1: $[-2, 2]$.
Julia set of f_2:



Ordinary rational maps

Julia set of f_3:



Definition

We say that a rational map f is *exceptional* if there exists an algebraic group G of dim. 1, an isogeny $\alpha : G \to G$, and a dominant map $\pi : G \to \mathbb{P}_1$ such that

$$f \circ \pi = \pi \circ \alpha.$$

Otherwise we call them ordinary.

Arithmetic of rational maps

We will now talk about arithmetic problems related to dynamical systems.

Heights

Let $h : \mathbb{P}_1 \to \mathbb{R}_{\geq 0}$ be the logarithmic Weil height. To each $f \in \overline{\mathbb{Q}}(z), \deg(f) \geq 2$ we can associate

$$\hat{h}_f : \mathbb{P}_1(\overline{\mathbb{Q}}) \to \mathbb{R}, \quad \hat{h}_f = \lim_{n\to\infty} \frac{h(f^{\circ n})}{d^n}$$

$$\{\hat{h}_f(z) = 0\} = \mathrm{Preper}(f) = \{z; |\{f^{\circ n}(z)\}_{n\geq 0}| < \infty\}.$$

Almost all $z \in \mathrm{Preper}(f)$ satisfy $z \in J(f)$.

There exists a measure μ_f of mass 1 on $\mathbb{P}_1(\mathbb{C})$, that satisfies $f^*\mu_f = d\mu_f$. Its support is $J(f)$.

Dynamical Bogomolov (Ghioca, Nguyen, Ye)

Let $C \subset \mathbb{P}_1^2$ be a curve and $f_1, f_2 \in \overline{\mathbb{Q}}(z), \deg(f_1) = \deg(f_2) \geq 2$ be ordinary. Then there exists $\epsilon, M > 0$

$$\{(z_1, z_2) \in C(\overline{\mathbb{Q}}); \hat{h}_{f_1}(z_1) + \hat{h}_{f_2}(z_2) < \epsilon\} \leq M$$

unless C is preperiodic. That is

$$|\{(f_1^{\circ n}, f_2^{\circ n})(C)\}| < \infty.$$

In their proof, both ϵ and M depend on C.

Families

We consider a function field of a curve $K = \overline{\mathbb{Q}}(B)$ and rational maps $f_1, f_2 \in K(z)$ of degree $d \geq 2$. On an open $B^0 \subset B$ holds that the *specializations* $f_{1,t}, f_{2,t} \in \overline{\mathbb{Q}}(z)$ are well-defined and have degree d, for $t \in B^0(\overline{\mathbb{Q}})$. For each $t \in B^0(\overline{\mathbb{Q}})$ we have a canonical height

$$\hat{h}_t : \mathbb{P}_1^2(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$$

given by $\hat{h}_t(z_1, z_2) = \hat{h}_{f_{1,t}}(z_1) + \hat{h}_{f_{2,t}}(z_2)$.

Uniform results

Families of curves

Let $C \subset \mathbb{P}_1^2$ be a curve defined over the function field K and dominating both factors $\mathbb{P}_1$. We consider it as a family $C \to B$ and denote a fibre by $C_t \subset \mathbb{P}_1^2$ (forgetting t).

Theorem (Mavraki, S.)

Suppose f_1, f_2 are ordinary. There exist constants $\epsilon > 0, M$ and an open $B' \subset B^0$ such that

$$|\{(z_1, z_2) \in C_t(\overline{\mathbb{Q}}); \hat{h}_t(z_1, z_2) < \epsilon\}| \leq M$$

for all $t \in B'(\overline{\mathbb{Q}})$ unless C is preperiodic by (f_1, f_2).

Comment: We prove that there are only finitely many fibres C_t that are pre-periodic if C is not pre-periodic. The condition on C to be dominant on both factors is necessary.

Families of polynomials

Uniform results for polynomials were obtained with different techniques by Demarco, Krieger and Ye.

Common pre-periodic points

Theorem (Demarco, Krieger and Ye)

There exists a constant $M = M(d)$ such that for all $t_1, t_2 \in \mathbb{C}$ holds that either

$$Preper(z^d + t_1) \cap Preper(z^d + t_2) \leq M$$

or $t_1 = t_2$.

This is a uniform Manin-Mumford theorem for the diagonal $\Delta \subset \mathbb{P}_1^2$ and the two dimensional base variety $\mathbb{A}^2$ (as opposed to a curve). Note that the set of parameters were Δ is pre-periodic forms a curve in $\mathbb{A}^2$. They also prove a statement for small heights instead of pre-periodic points with a uniform ϵ.

Proofs and going further

The proof of Mavraki and me serves as a blue-print for further progress. We use equi-distribution results, recently published by Yuan and Zhang, and a local Hodge index theorem. Our proof goes via proving a relative Bogomolov conjecture à la Kühne. With our proof strategy and some more input one can go towards higher dimensional bases. A conjecture for higher dimensional bases is:

Conjecture (Demarco, Krieger, Ye)

For all $d \geq 2$ there exists a constant $M = M(d)$ such that for all $f_1, f_2 \in \mathbb{C}(z)$ of degree d holds

$$|Perper(f_1) \cap Preper(f_2)| \leq M$$

or $Perper(f_1) = Preper(f_2)$.

Theorem (WIP)

Let $f \in K[z]$ be a family of ordinary polynomials of degree $d \geq 2$ over a base curve B ($K = \overline{\mathbb{Q}}(B)$) such that each specialization is a polynomial of degree d. There exists a constant $M = M(f, B)$ such that for all $t_1, t_2 \in B(\mathbb{C})$ either

$$|Preper(f_{t_1}) \cap Preper(f_{t_2})| \leq M$$

or

$$Preper(f_{t_1}) = Preper(f_{t_2}).$$

Comment: This follows from a relative Bogomolov theorem over a 2 dimensional base and the proof uses Böttcher coordinates. We also show that the set of $(t_1, t_2) \in B^2(\mathbb{C})$ that satisfies $\text{Preper}(f_{t_1}) = \text{Preper}(f_{t_2})$ forms a finite union of subvarieties of B^2.

Thank you!