Algebraic Matroids + Tropical Geometry

Throughout: E is a finite set

$\mathbb{C}^E$ has coordinates indexed by E

$\pi_S : \mathbb{C}^E \to \mathbb{C}^S$ coordinate proj $\forall\, S \subseteq E$.

Varieties are irreducible and come of cardinals(?) i.e. live in $\mathbb{C}^E$, fixed basis indexed by E.

<u>Defn:</u> The algebraic matroid of $V \subseteq \mathbb{C}^E$, $\mathcal{M}(V)$, is the set of $S \subseteq E$ such that $\dim(\pi_S(V)) = \dim(V)$. Called "spanning sets"

<u>Ex:</u> Define $V \subseteq \mathbb{C}^{\{1,2,3\}}$ by

$V = \{(x_1,x_2,x_3) \in \mathbb{C}^3 : x_3 = 0\}$

Spanning sets are all subsets of $\{1,2,3\}$ that include 1 or 2.

<u>Ex:</u> Define $\varphi : (\mathbb{C}^2)^5 \to \mathbb{C}^{E(K_5)} = \mathbb{R}^{\binom{5}{2}}$ by

$$(x_1, y_1, \dots, x_5, y_5) \mapsto \left((x_i - x_j)^2 + (y_i - y_j)^2\right)_{1 \le i < j \le 5}$$

Define $V = \overline{\mathrm{im}(\varphi)}$.

Note: subsets of $E(K_5)$ are graphs on 5 vertices

The minimal spanning sets (AKA bases) are

These are the minimally generically rigid graphs on 5 vertices in $\mathbb{R}^2$.

<u>Ex:</u> Cayley-Menger varieties: CM_n^d

Generalize above to arbitrary dimension d and number of points n.

<u>Fundamental Problem of Rigidity Theory:</u>

Find a "nice" description of the matroid of CM_n^d. Equiv: Which graphs are rigid in d-dimensions?

<u>Thm:</u> G is spanning in CM_n^1 iff G is connected.

<u>Thm</u> (Hilda Geiringer 1927) G is a basis of CM_n^2 iff G has $2n-3$ edges and every subgraph on m vertices has at *most* $2m-3$ edges.

Major open problem: CM_n^3

<u>Ex:</u> Algebraic matroids of determinantal varieties have relevance to low-rank matrix completion. Talk to me for details.

<u>Ex:</u> Certain identifiability questions can be phrased as asking for the algebraic matroid of some variety

---

<u>Tropical Geometry:</u>

Let $V \subseteq \mathbb{C}^E$ be a variety defined by ideal $I \subseteq \mathbb{C}[E] = \mathbb{C}[x_e : e \in E]$

Each $w \in \mathbb{R}^E$ defines a deformation of V as follows

- For each $f = \sum_{\alpha \in \Omega} c_\alpha x^\alpha \in \mathbb{C}[E]$ define $\mathrm{in}_w(f) := \sum_{\alpha \in \mathrm{argmin}\{w \cdot \alpha\}} c_\alpha x^\alpha$
- Define $\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) : f \in I \rangle$

Then $V(\mathrm{in}_w(I))$ is a Gröbner degeneration of V.

The tropicalization of V is

$$\mathrm{trop}(V) := \{w \in \mathbb{R}^E : \mathrm{in}_w(I(V)) \text{ has no monomials}\}$$

<u>Thm (many authors, multiple papers)</u>

Let $V \subseteq \mathbb{C}^E$ be a variety of dimension d. Then $\mathrm{trop}(V)$ is a highly structured polyhedral fan of pure dimension d.

<u>Ex:</u> Let $L \subseteq \mathbb{C}^E$ be a linear space. then $\mathrm{trop}(L)$ depends only on the (algebraic) matroid of L. In particular, the cones are indexed by flags of flats of M.

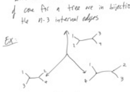
The cone corresponding to $\emptyset = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k$ is spanned by the characteristic vectors of F_i's and $\pm(1, \dots, 1)$.

<u>Cor (Josephine Yu):</u> $S \subseteq E$ is a spanning set of $V \subseteq \mathbb{C}^E$ iff $\dim(\pi_S(\mathrm{trop}(V))) = \dim(V)$

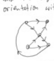
<u>Thm (Develin + Sturmfels c. 2008):</u>

Let $Gr_{2,n} \subseteq \mathbb{C}^{\binom{n}{2}}$ be the (affine cone over) the Grassmannian of 2-planes in $\mathbb{C}^n$. Then $\mathrm{trop}(Gr_{2,n})$ has a polyhedral structure whose maximal cones are indexed by binary leaf-labeled trees on n-leaves. They all have a common lineality space of dimension n. Modulo this, the rays of cone for a tree are in bijection with the $n-3$ internal edges.

<u>Ex:</u>



<u>Thm</u> (B. c. 2017) G is a minimal spanning set in $Gr_{2,n}$ iff G has $2n-3$ edges and the edges of G have an acyclic orientation with no alternating closed trails.



bad orientation, but can be fixed

<u>Remark:</u> If $W \subseteq \mathbb{C}^E$ is a Gröbner degeneration of $V \subseteq \mathbb{C}^E$ and $S \subseteq E$ is spanning in $\mathcal{M}(W)$, then it is spanning in $\mathcal{M}(V)$. I believe that there should be some kind of converse, i.e. $S \subseteq E$ spanning in $\mathcal{M}(V)$ implies there is a matroid Gröbner degeneration W of V, not contained in a union of coordinate subspaces, such that S is spanning in $\mathcal{M}(W)$.

<u>Next steps:</u> use tropical methods to give general tools for understanding these matroids

---

<u>Hadamard Products of Varieties:</u>

Given $U, V \subseteq \mathbb{C}^E$, define the Hadamard product

$$U * V = \{(u_e v_e)_{e \in E} : u \in U, v \in V\}$$

<u>Goal:</u> write down a formula for $\mathcal{M}(U * V)$ in terms of $\mathcal{M}(U)$ and $\mathcal{M}(V)$ when U and V are linear spaces.

<u>Reason:</u> Many varieties in 2d rigidity can be expressed this way, e.g.

<u>Defn:</u> A <u>matroid</u> is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I} \subseteq 2^E$ satisfies

1) $\emptyset \in \mathcal{I}$
2) $I \in \mathcal{I}$ and $J \subseteq I \Rightarrow J \in \mathcal{I}$
3) If $I, J \in \mathcal{I}$ and $|J| > |I|$ then $\exists\, x \in J \setminus I$ st $I \cup \{x\} \in \mathcal{I}$

The rank function of a matroid $(E, \mathcal{I})$ is

$$\rho : 2^E \to \mathbb{Z}, \quad S \mapsto \max_{\substack{I \in \mathcal{I} \\ I \subseteq S}} |I|$$

<u>Thm</u> (Edmonds–Rota 1966) Given a finite set E and a function $f : 2^E \to \mathbb{Z}$, define

$$\mathcal{I} = \{I \subseteq E : |I'| \le f(I') \ \forall\, I' \subseteq I\}$$

Then $(E, \mathcal{I})$ is a matroid, denoted $\mathcal{M}(f)$, if f satisfies:

1) $f(S) \ge 0$ if $S \ne \emptyset$
2) $T \subseteq S \Rightarrow f(T) \le f(S)$
3) $f(S \cup T) + f(S \cap T) \le f(S) + f(T)$.

<u>Thm</u> (B. 2021) Let U, V be linear spaces and let ρ_u, ρ_v be the rank functions of $\mathcal{M}(U)$ and $\mathcal{M}(V)$. Then

$$\mathcal{M}(U * V) = \mathcal{M}(\rho_u + \rho_v - 1).$$

<u>Directions to Generalize:</u>

1) More general varieties. The proof still goes through if $\exists$ linear spaces L_1, L_2 such that $\mathrm{trop}(U) = L_1$, $\mathrm{trop}(V) = L_2$. Caution: formula fails in general for e.g. toric varieties
2) More than just 2. Conjecture: given linear spaces $L_1, \dots, L_k$, $\mathcal{M}(L_1 * \dots * L_k)$ is $\mathcal{M}(\rho_1 + \dots + \rho_k - k + 1)$
3) More general quadratic monomial images of linear spaces.

Relevant papers:

"Generic symmetry-forced infinitesimal rigidity: translations and rotations"

"Completion of tree metrics and rank two matrices"