Global well-posedness for the derivative nonlinear Schrödinger equation on the line

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joint work with B. Harrop-Griffiths, R. Killip and M. Visan

The derivative nonlinear Schrödinger equation

$$iq_t + q'' + i\left(|q|^2 q\right)' = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R} \qquad \text{(DNLS)}$$
$$q(0,x) = q_0(x) \in H^s(\mathbb{R})$$

It arises as a model for the propagation of large-wavelength Alfvén waves in plasma.

Well-posedness

-Existence of solution
-Uniqueness of solution
-Continuous dependence on initial data

Getting started

- (DNLS) enjoys the scaling symmetry

$$q(t,x) \mapsto q_\lambda(t,x) = \sqrt{\lambda}\, q(\lambda^2 t, \lambda x).$$

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  The L^2 norm is preserved under the scaling. *L^2-critical*

- The L^2 norm is also preserved under the flow.
  In fact, (DNLS) has an infinite family of conserved quantities:

$$
\begin{aligned}
M(q) &= \int |q|^2 \, dx \\
H(q) &= -\tfrac{1}{2} \int i(q\bar{q}' - \bar{q}q') + |q|^4 \, dx \\
H_2(q) &= \int |q'|^2 + \tfrac{3}{4} i |q|^2 (q\bar{q}' - \bar{q}q') + \tfrac{1}{2}|q|^6 \, dx \\
&\vdots
\end{aligned}
$$

  It is *completely integrable*.

Previous results

- Local well-posedness: in H^s, $s \geq \frac{1}{2}$ (Tsutsumi–Fukuda '81, Takaoka '99)
- Ill-posedness: uniformly continuous dependence on initial data breaks down below $s = \frac{1}{2}$ (Biagioni–Linares '01)

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From Local to Global

The time of existence given by the local well-posedness results above depends on $\|q_0\|_{H^{\frac{1}{2}}}$.

One would hope that the conservation laws of (DNLS) could be exploited to obtain global well-posedness.

Problem: lack of coercivity!

Previous results: Global well-posedness

- Under the assumption that $\|q_0\|_{L^2}^2 < 2\pi$
  In H^1: Hayashi–Ozawa '92
  In H^s, $s > \frac{1}{2}$: Colliander–Keel–Staffilani–Takaoka–Tao '02
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$4\pi = M(q_a)$ for the algebraic soliton, $q_a(t,x) = e^{it/4} q_0(x - t)$.
It is the threshold where the conservation laws lose their efficacy.
Rescaling the algebraic soliton, we get a family of solutions

$$q_{a,\lambda}(t,x) = \sqrt{\lambda}\, q_a(\lambda^2 t, \lambda x), \quad \lambda > 0$$

that have the same values for all the conserved quantities, but is unbounded in H^s for all $s > 0$!

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- Under no L^2 norm assumptions
  In $H^{2,2} = \{f \in H^2 : x^2 f \in L^2\}$: Jenkins–Liu–Perry–Sulem '20
  In $H^{1,1} \cap H^2$: Pelinovsky–Saalmann–Shimabukuro '17

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  In $H^{\frac{1}{2}}$: Bahouri–Perelman '22

Our main results

The enemy is concentration of the L^2 norm in one or more locations in space.
The key is an equicontinuity result.
$Q \subset L^2$ is *equicontinuous* if

$$q(x+h) \to q(x) \quad \text{in } L^2 \text{ as } h \to 0, \text{ uniformly for } q \in Q.$$

Theorem (Killip–N.–Visan)

Let $Q \subseteq L^2$ be an equicontinuous set satisfying

$$\sup\{\|q_0\|_{L^2}^2 : q_0 \in Q\} < 4\pi.$$

Then the totality of states reached by (DNLS) *orbits originating from Q*

$$Q_* = \{e^{tJ\nabla H} q_0 : q_0 \in Q \text{ and } t \in \mathbb{R}\}$$

is also L^2-equicontinuous.

Our main results

Theorem (Killip–N.–Visan)

Fix $0 < s < \frac{1}{2}$ *and let* Q *be a bounded subset of* H^s *satisfying*

$$\sup\left\{\|q_0\|_{L^2}^2 : q_0 \in Q\right\} < 4\pi.$$

Then

$$\sup_{q \in Q_*} \|q\|_{H^s} \lesssim C\Big(\sup_{q_0 \in Q} \|q_0\|_{L^2}^2, \sup_{q_0 \in Q} \|q_0\|_{H^s}^2\Big)$$

Moreover, if Q *is* H^s*-equicontinuous, then so is* Q_*.

Theorem (Killip–N.–Visan)

Fix $\frac{1}{6} \leq s < \frac{1}{2}$. *The* (DNLS) *evolution is globally well-posed for all*

$$q_0 \in H^s \quad \text{with} \quad \|q_0\|_{L^2}^2 < 4\pi.$$

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Corollary
Fix $\frac{1}{6} \leq s < \frac{1}{2}$. The (DNLS) *evolution is globally well-posed in H^s.*

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Theorem (Harrop-Griffiths–Killip–N.–Visan)
The (DNLS) *evolution is globally well-posed in L^2.*

Complete Integrability

- Infinitely many conservation laws:

$$M(q) = \int |q|^2 \, dx$$

$$H(q) = -\tfrac{1}{2} \int i(q\bar{q}' - \bar{q}q') + |q|^4 \, dx$$

$$\vdots$$

- Lax pair

$$L(q(t)), P(q(t))$$

$$\frac{\partial}{\partial t} L(t) = [P(t), L(t)] \iff q(t) \text{ solution}$$

"The spectral properties of L are preserved under the flow."

- Inverse Scattering Transform

Lax operator and perturbation determinant

$$L(\kappa) = \begin{bmatrix} \kappa - \partial & \sqrt{\kappa} q \\ -i\sqrt{\kappa}\bar{q} & -(\kappa + \partial) \end{bmatrix}, \qquad \kappa \geq 1$$

$$L_0(\kappa) = \begin{bmatrix} \kappa - \partial & 0 \\ 0 & -(\kappa + \partial) \end{bmatrix} \qquad (\text{for } q \equiv 0).$$

Perturbation determinant:

$$\det\left[L_0^{-1}(\kappa)L(\kappa)\right] \rightsquigarrow \cdots \rightsquigarrow \det\left[1 - i\kappa\Lambda\Gamma\right] := a(\kappa; q),$$

$$\Lambda(\kappa; q) := (\kappa - \partial)^{-\frac{1}{2}} q (\kappa + \partial)^{-\frac{1}{2}}, \; \Gamma(\kappa; q) := (\kappa + \partial)^{-\frac{1}{2}} \bar{q} (\kappa - \partial)^{-\frac{1}{2}}.$$

We also consider

$$\alpha(\kappa; q) := -\log\det(1 - i\kappa\Lambda\Gamma) = \sum_{\ell \geq 1} \tfrac{1}{\ell} \operatorname{tr}\left\{(i\kappa\Lambda\Gamma)^{\ell}\right\}$$

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Preserved under (DNLS) (Klaus–Schippa)

Equicontinuity and α

Q is *equicontinuous* in L^2 if

$$
\begin{aligned}
&q(x+h) \to q(x) \quad \text{in } L^2 \text{ as } h \to 0, \text{ uniformly for } q \in Q \\
\iff &\int_{|\xi|>N} |\hat{q}(\xi)|^2 d\xi \to 0 \quad \text{as } N \to \infty, \text{ uniformly for } q \in Q
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$\alpha(\kappa; q) = \sum_{\ell \geq 1} \frac{1}{\ell} \operatorname{tr}\left\{(i\kappa\Lambda\Gamma)^\ell\right\}$

- $\alpha(\kappa; q)$ is conserved under (DNLS)
- For $\ell = 1$: $\operatorname{Im}\operatorname{tr}(i\kappa\Lambda\Gamma) = \int_{\mathbb{R}} \frac{2\kappa^2}{4\kappa^2+\xi^2} |\hat{q}(\xi)|^2 d\xi$

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- The quadratic term of $\beta(\kappa)$ is $\beta^{[2]}(\kappa; q) = \int_{\mathbb{R}} \frac{\xi^2}{4\kappa^2+\xi^2} |\hat{q}(\xi)|^2 d\xi$.

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Q is equicontinuous in L^2

$\iff \beta^{[2]}(\kappa; q) \to 0$ as $\kappa \to \infty$ uniformly in Q

Commuting flows

$\alpha(\kappa; q)$ admits the asymptotic expansion

$$\alpha(\kappa; q) = \frac{i}{2} M(q) + \frac{1}{4\kappa} H(q) + O(\kappa^{-2})$$

This leads to

$$H_\kappa(q) := 4\kappa \mathrm{Re}\alpha(\kappa; q) = H(q) + O(\kappa^{-1}).$$

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- H and H_κ define commuting flows.
- The equicontinuity and the H^s bounds results discussed above also hold for the H_κ flows.
- It is fairly easy to prove that they are well-posed in H^s, $s \geq 0$.
- The hard part is showing that the H_κ flows converge to (DNLS) in H^s as $\kappa \to \infty$. This is where the assumption $s \geq \frac{1}{6}$ becomes necessary!

Towards critical well-posedness

We saw that the H_κ flows are globally well-posed in L^2.
All we need to adapt our commuting flows argument in the L^2 setting is the convergence of the H_κ flows in L^2.

Problem

This would require additional regularity on q...

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Solution

Local smoothing!

"Locally in space, on average in time, we can gain half a derivative."

Local smoothing

Theorem (Local smoothing for (DNLS))

Let $Q \subset L^2(\mathbb{R})$ be bounded and equicontinuous and $\psi \in \mathcal{S}(\mathbb{R})$. Then for each $T > 0$ solutions to (DNLS) *with initial data $q(0) \in Q$ satisfy*

$$\int_{-T}^{T} \|\psi q(t)\|^2_{H^{\frac{1}{2}}(\mathbb{R})} dt \lesssim_{Q,T} \|q(0)\|^2_{L^2}.$$

Theorem (Local smoothing for the difference flow)

Let $Q \subset L^2(\mathbb{R})$ be bounded and equicontinuous and $\psi \in \mathcal{S}(\mathbb{R})$. Then for each $T > 0$ solutions to the flow induced by $H - H_\kappa$ with initial data $q(0) \in Q$ satisfy

$$\int_{-T}^{T} \|\psi q(t)\|^2_{H^{\frac{1}{2}}(\mathbb{R})} dt \lesssim_{Q,T} \|q(0)\|^2_{L^2}.$$

Local smoothing

Equicontinuity is important!

Rescaling a stationary soliton $q_s(t,x) = e^{it} q_0(x)$, we get a family of solutions $q_{s,\lambda}(t,x)$ with the same L^2 norm, but

$$\int_{-1}^{1} \|\psi(x) q_{s,\lambda}(t,x)\|^2_{H_x^{\frac{1}{2}}(\mathbb{R})} \, dt \approx \lambda.$$

L^2 boundedness alone is not enough to control the local smoothing norm.

Local smoothing

Idea

To prove local smoothing, we rely once again on the conservation of $\alpha(\varkappa; q)$.
We can write $\alpha(\varkappa; q) = \int \rho(\varkappa; q, x)\, dx$ and we have the microscopic conservation laws

$$\partial_t \rho(\varkappa) + \partial_x j_{DNLS}(\varkappa) = 0 \qquad \text{for DNLS,}$$
$$\partial_t \rho(\varkappa) + \partial_x j_{H-H_\kappa}(\varkappa) = 0 \qquad \text{for the difference flow.}$$

Then

$$\int_{-T}^{T} \int j(\varkappa; q(t), x)\psi(x)\, dx\, dt = \int \rho(\varkappa; q(t), x)\tilde{\psi}(x)\, dx \Big|_{-T}^{T}.$$

Although the general idea is the same for both flows, proving local smoothing for the difference flow is much harder.

Thank you!