The soliton problem for the Zakharov Water-Waves system with a slowly varying bottom

Joint work with Claudio Muñoz and Frédéric Rousset

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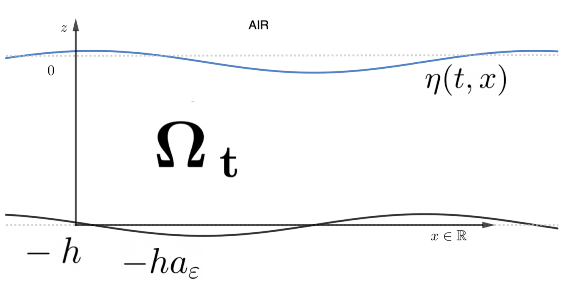
New trends in Mathematics of Dispersive, Integrable and Nonintegrable Models in Fluids, Waves and Quantum Physics

Introduction

Mathematical formulation

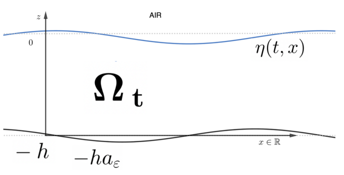
We consider a **fluid** contained in a domain with **rigid bottom** and **free surface** that separates it from vacuum, Ω_t.



$$\Omega_t = \left\{(x, z) \in \mathbb{R}^2 \text{ such that } ha_\varepsilon(x) \leqslant z \leqslant \eta(t, x)\right\},$$

$0 < h < \infty$ is the constant reference depth, $\varepsilon > 0$, t is time and $\eta(t, x) \in \mathbb{R}$ is the (unknown) free surface elevation.

Mathematical formulation



The following assumptions are made on the fluid and on the flow:

(H1) The fluid is homogeneous and inviscid.

(H2) The fluid is incompressible.

(H3) The flow is irrotational.

(H4) The surface and the bottom can be parametrized as graphs above the still water level.

(H5) The fluid particles do not cross the bottom.

(H6) The fluid particles do not cross the surface.

(H7) There is surface tension.

(H8) The fluid is at rest at infinity.

(H9) The water depth is always bounded from below by a nonnegative constant.

Mathematical formulation

We denote **u** the **velocity of the fluid**, and there exists a scalar function Φ such that inside the fluid domain Ω_t,

$$\mathbf{u} = (\partial_x \Phi, \partial_z \Phi) = \nabla_{x,z}\Phi \quad \text{in } \Omega.$$

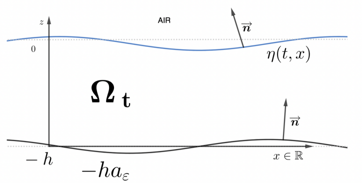
$$\text{(H1)} \quad \partial_t \Phi + \frac{1}{2}|\nabla_{x,z}\Phi|^2 + gz = -\frac{1}{\rho}(P - P_{atm}) \qquad \text{in } \Omega_t$$

$$\text{(H2)} \quad \Delta_{x,z}\Phi = 0 \qquad (\nabla \cdot \mathbf{u} = 0) \qquad \text{in } \Omega_t$$

$$\text{(H5)} \quad \partial_{\mathbf{n}}\Phi = 0 \qquad (\mathbf{u} \cdot n = 0) \qquad \text{on } \{z = -ha_\varepsilon(x)\}$$

$$\text{(H6)} \quad \partial_t \eta - \sqrt{1 + |\partial_x \eta|^2}\, \partial_{\mathbf{n}}\Phi = 0 \qquad \text{on } \{z = \eta(t,x)\}$$

$$\text{(H7)} \quad \frac{1}{\rho}(P - P_{atm}) = -\beta \nabla \cdot \left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \qquad \text{on } \{z = \eta(t,x)\}$$



Mathematical formulation

The Zakharov water-waves problem arises when noticing that

$$(\eta(t,x), \varphi(t,x)) := (\eta(t,x), \Phi(t,x,\eta(t,x)))$$

fully determine the flow.

We define the **Dirichlet-Neumann operator**, first introduced by Craig-Sulem-Sulem:

$$\mathcal{G}[\eta, a] : \varphi \mapsto \sqrt{1+|\nabla\eta|^2}\, \partial_n \Phi\Big|_{z=\eta}$$

How to recover the the velocity potential $\Phi(t,\cdot,\cdot)$?
$\Phi(t,\cdot,\cdot)$ is the solution to the equation

$$\begin{cases} \Delta_{x,z}\Phi = 0 & (x,z) \in \Omega_t \\ \Phi\Big|_{z=\eta} = \varphi & \\ \partial_{\mathbf{n}}\Phi\Big|_{z=-ha_\varepsilon} = 0. & \end{cases}$$

Mathematical formulation

In consequence, the one-dimensional Zakharov Water-Waves (ZWW) problem can be written as

$$\begin{cases} \partial_t \eta = \mathcal{G}[\eta, a]\varphi \\ \partial_t \varphi = -\frac{1}{2}\left|\partial_x \varphi\right|^2 + \frac{1}{2}\frac{\left(\mathcal{G}[\eta, a]\varphi + \partial_x \varphi \partial_x \eta\right)^2}{1 + \left|\partial_x \eta\right|^2} - g\eta + \beta \partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + \left|\partial_x \eta\right|^2}}\right) \end{cases}$$

where g is the gravitational constant and β is the tension surface coefficient.

Denoting $\mathbf{U} = (\eta, \varphi)^T$, we can write (ZWW) in the abstract form

$$\partial_t \mathbf{U} = \mathcal{F}(\mathbf{U})$$

where

$$\mathcal{F}(\mathbf{U}) = \begin{pmatrix} \mathcal{G}[\eta, a]\varphi \\ -\frac{1}{2}\left|\partial_x \varphi\right|^2 + \frac{1}{2}\frac{\left(\mathcal{G}[\eta, a]\varphi + \partial_x \varphi \partial_x \eta\right)^2}{1 + \left|\partial_x \eta\right|^2} - g\eta + \beta \partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + \left|\partial_x \eta\right|^2}}\right) \end{pmatrix}$$

About the equation

ZWW system has a Hamiltonian structure in the variable (η, φ):

$$\partial_t \begin{pmatrix} \eta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\eta \mathcal{H} \\ \partial_\varphi \mathcal{H} \end{pmatrix}$$

where the Hamiltonian $\mathcal{H}$ is the total energy given by

$$\mathcal{H}(\eta, \varphi) = \frac{1}{2} \int_{\mathbb{R}^2} \varphi \mathcal{G}[\eta, a] \varphi + g\eta^2 + 2\beta \left(\sqrt{1 + |\nabla \eta|^2} - 1 \right) dxdz.$$

Well-posedeness:

$(\beta = 0)$ Global existence for dimensions $d = 2, 3$.
Wu (2009), Wu (2010), Germain-Masmoudi-Shatah (2009)

$(\beta \neq 0)$ Local existence in $H^{s+1/2} \times H^s$, $s > 5/2$ for $d = 2$.
Germain-Masmoudi-Shatah (2012), Alazard-Burq-Zuily (2009).

About solitary waves

Solitons exist for the flat-bottom problem.

Existence of solitary waves

Theorem (Amick-Kirchgässner)

Suppose that g, β, h *satisfy*

$$\frac{gh}{c^2} = 1 + \epsilon^2, \quad \frac{\beta}{hc^2} > \frac{1}{3}.$$

Then, there exists ϵ_0 *such that for every* $\epsilon \in (0, \epsilon_0)$*, there exists a solution of (ZWW) with flat-bottom*

$$\boldsymbol{Q}_c(x - ct) = (\eta_c(x - ct), \varphi_c(x - ct))$$

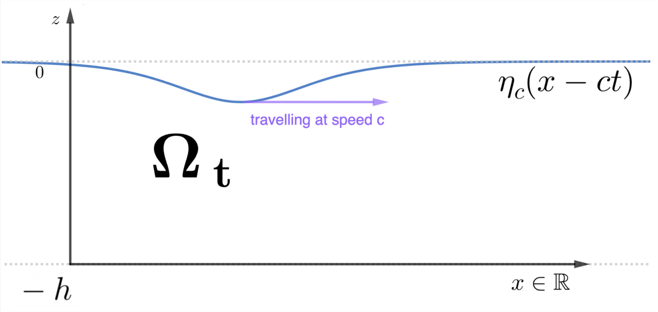
with

$$\eta_c(x) = h\epsilon^2 \Theta_1(\epsilon h^{-1} x, \epsilon) \quad \varphi_c(x) = ch\epsilon\Theta_2(\epsilon h^{-1} x, \epsilon)$$

where Θ_1 *is even,* Θ_2 *is odd and satisfy an exponential decay.*

About solitary waves

Solitons exist for the flat-bottom problem. We have existence of solitary waves of the form $Q_c(x-ct) = (\eta_c(x-ct), \varphi_c(x-ct))$ of speed $c \sim \sqrt{gh}$



And satisfy

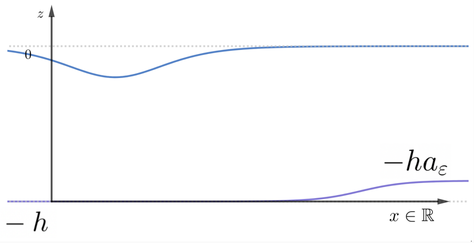
$$\exists d > 0, \quad \forall \alpha \geqslant 0, \quad \exists C_\alpha > 0, \quad \forall (x, \epsilon) \in \mathbb{R} \times (0, \epsilon_0), \quad |\partial^\alpha_{x,t} \eta_c| \leqslant C_\alpha e^{-d|x-ct|}$$

$$\exists d > 0, \quad \forall \alpha \geqslant 1, \quad \exists C_\alpha > 0, \quad \forall (x, \epsilon) \in \mathbb{R} \times (0, \epsilon_0), \quad |\partial^\alpha_{x,t} \varphi_c| \leqslant C_\alpha e^{-d|x-ct|}.$$

Main goal:

We want to construct and describe the solitary wave-type solution for the non-flat bottom problem.

A sketch of the situation:



In mathematical terms, we want to prove the existence of a solution

$$\begin{pmatrix} \eta \\ \varphi \end{pmatrix} \to \begin{pmatrix} \eta_c \\ \varphi_c \end{pmatrix} (x + A - ct) \quad \text{as} \quad t \to -\infty.$$

where $A \in \mathbb{R}$ is such that $A \gg 1$, and $\mathbf{Q}_c = (\eta_c, \varphi_c)^t$ is a solitary wave of the flat-bottom problem (we know it exists thanks to Amick-Kirchgässner).

(Ming-Rousset-Tzvetkov, Martel.)

Goal

Hypothesis on the bottom

We consider $a_\varepsilon(x) = a(\varepsilon x)$ for $\varepsilon > 0$ sufficiently small, where $a \in C^2(\mathbb{R})$ satisfies: There exist $K > 0$, $\kappa \in \mathbb{R}$, $|\kappa| < 1$ and $\gamma > 0$ such that

$$a \text{ non-increasing or non-decreasing},$$

$$1 < a(r) < 1 + \kappa \quad \forall r \in \mathbb{R},$$

$$|a'(r)| < Ke^{-\gamma|r|} \quad \forall r \in \mathbb{R},$$

$$\lim_{r\to-\infty} a(r) = 1, \ \lim_{r\to\infty} a(r) = 1 + \kappa.$$

Let us fix $s \geqslant 0$. Suppose the existence of a solitary wave $\mathbf{Q}_c$ of speed $c \geqslant 0$ in the flat-bottom system.

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Theorem (M. - Muñoz - Rousset, 2021)

There exists $\varepsilon^* > 0$ such that, for every $\varepsilon \in (0, \varepsilon^*)$ and A large enough, there exists a solution $\boldsymbol{U} = (\eta, \varphi)^t$ to (ZWW), that satisfies

$$\boldsymbol{U} - \boldsymbol{Q}_c \in \mathcal{C}_b(\mathbb{R}_-, H^s(\mathbb{R}) \times H^s(\mathbb{R})), \text{ and}$$

$$\lim_{t \to -\infty} \|\boldsymbol{U}(t) - \boldsymbol{Q}_c(\cdot - ct + A)\|_{H^s(\mathbb{R})} = 0.$$

Definition:

We define for $\boldsymbol{U} = (\eta, \varphi)^t$, *the* $|\cdot|_{E^s}$, *for* $s \geqslant 0$, *as*

$$|\boldsymbol{U}|_{E^s} = \sum_{|\alpha| \leqslant s} |\partial_{t,x}^{\alpha} \boldsymbol{U}|_{L^2}.$$

Step 1: Plugging the solitary wave into the non-flat bottom problem.

Since $\mathbf{Q}_c = (\eta_c, \varphi_c)^t$ solves the *flat bottom problem* $(a = 1)$, we have that

$$\begin{cases} \partial_t \eta_c = \mathcal{G}[\eta_c, a_\varepsilon]\varphi_c + r_1(a) \\ \partial_t \varphi_c = -\frac{1}{2}|\partial_x \varphi_c|^2 + \frac{1}{2}\frac{(\mathcal{G}[\eta, 1]\varphi_c + \partial_x \varphi_c \partial_x \eta_c)^2}{1 + |\partial_x \eta_c|^2} - g\eta_c + b\partial_x \left(\frac{\partial_x \eta_c}{\sqrt{1 + |\partial_x \eta_c|^2}} \right) + r_2(a) \end{cases}$$

for

$$r_1(a) = \mathcal{G}[\eta_c, 1]\varphi_c - \mathcal{G}[\eta_c, a_\varepsilon]\varphi_c$$

$$r_2(a) = \frac{1}{2}\frac{r_1(a)\left(\mathcal{G}[\eta_c, 1]\varphi_c + \mathcal{G}[\eta_c, a_\varepsilon]\varphi_c\right) + 2r_1(a)\partial_x \varphi_c \partial_x \eta_c}{1 + |\partial_x \eta_c|^2}.$$

Exponential decay of the reminder

Proposition:

The remainder $\boldsymbol{r}(a) = (r_1(a), r_2(a))^t$ *has an exponential decay in time. That is, there exist* $0 < \delta_0 < \min\{\gamma\varepsilon, \delta\}$ *and* $C_s > 0$ *such that for every* $s \geqslant 0$,

$$|\boldsymbol{r}(a)|_{E^s} \leqslant C_s e^{-\delta_0 A} e^{\delta_0 ct}, \quad \textit{for all } t \leqslant 0.$$

Step 2: Construction of an aproximate solution. Ansatz.

Goal: To construct an **aproximate solution** $\mathbf{U}_{ap} = \mathbf{Q}_c + \mathbf{V} \to \mathbf{Q}_c$ in the sense that

$$\partial_t \mathbf{U}_{ap} = \mathcal{F}(\mathbf{U}_{ap}) + \mathbf{r}_{ap},$$

where $\mathbf{r}_{ap} \to 0$ (hopefully, with exponential rate).

Ansatz: Take $\rho = e^{-\delta_0 A} \Rightarrow \rho \ll 1$ if $A \gg 1$ and $\mathbf{r} = \rho \mathbf{r}_c$ with

$$|\mathbf{r}_c|_{H^s} \leqslant Ce^{\delta_0 ct}, \text{ for } t \leqslant 0.$$

We define

$$\mathbf{V}(t,x) = \sum_{l=1}^{N} \rho^l \mathbf{V}_l(t,x),$$

for $\mathbf{V}_l$ still unknown (to be constructed) and $N > 0$ sufficiently large. If we make Taylor expansion of $\mathcal{F}$ around the solitary wave, we have that

$$\mathcal{F}(\mathbf{U}_{ap}) = \mathcal{F}(\mathbf{Q}_c + \mathbf{V}) = \mathcal{F}(\mathbf{Q}_c) + \sum_{j=1}^{N} \frac{1}{j!} D^j \mathcal{F}[\mathbf{Q}_c](\mathbf{V}, \dots, \mathbf{V}) + \mathbf{r}_{N,\gamma}(V)$$

Sketch of proof - Step 2

Using (ZWW), we obtain a linear problem for each $\mathbf{V}_l$:

Equation for $\mathbf{V}_1$:

$$\partial_t \mathbf{V}_1 - D\mathcal{F}[\mathbf{Q}_c]\mathbf{V}_1 = -\mathbf{r}_c.$$

Equation for V_2:

$$\partial_t \mathbf{V}_2 - D\mathcal{F}[\mathbf{Q}_c]\mathbf{V}_2 = \frac{1}{2} D^2\mathcal{F}[\mathbf{Q}_c](\mathbf{V}_1, \mathbf{V}_1)$$

Equation for any $\mathbf{V}_j$, $j \in \{2, \dots N\}$:

$$\partial_t \mathbf{V}_j - D\mathcal{F}[\mathbf{Q}_c]\mathbf{V}_j = \sum_{p=1}^{j} \sum_{\substack{1 \leqslant j_1, \dots, j_p \leqslant j-1 \\ j_1 + \dots j_p = j}} \frac{1}{p!} D^p \mathcal{F}[\mathbf{Q}_c](\mathbf{V}_{j_1}, \dots, \mathbf{V}_{j_p})$$

We need to study the homogeneous linear system!

Step 3: Analysis of the linear system

We consider the homogeneous linear equation

$$\partial_t \mathbf{V} - D\mathcal{F}[\mathbf{Q}_c]\mathbf{V} = 0 \tag{1}$$

which corresponds to the **linearization of the ZWW system about the solitary wave $\mathbf{Q}_c$**.

How fast does the fundamental solution of (1) grow?

Define S_Q^Λ the fundamental solution to (1).

Growth of the fundamental solution

Theorem:

Asumme that the solitary wave exists. Then, for any $k \geqslant 0$, there exists A_0 such that for every $A \geqslant A_0$,

$$|S_Q^\Lambda(t,\tau)\boldsymbol{V}|_{E^k} \leqslant A^{1/4}|\boldsymbol{V}|_{H^{s(k)}}(1+|t-\tau|)e^{\delta_0 c|t-\tau|/2}.$$

Sketch of proof - Step 4

Step 4: Construction of the aproximate solution. Decay rates.

Recall the definiton

$$\mathbf{V}(t,x) = \sum_{l=1}^{N} \rho^l \mathbf{V}_l(t,x)$$

For instance, for $\mathbf{V}_1$,

$$\partial_t \mathbf{V}_1 - D\mathcal{F}[\mathbf{Q}_c]\mathbf{V}_1 = -\mathbf{r}_c$$

where,

$$|\mathbf{r}_c|_{H^s} \leqslant Ce^{\delta_0 ct}, \text{ for } t \leqslant 0, \ s \in \mathbb{N},$$

and also

$$|S_Q^\Lambda(t,\tau)\mathbf{V}|_{E^k} \leqslant CA^{1/4}|\mathbf{V}|_{H^{s(k)}}(1+|t-\tau|)e^{\delta_0 c|t-\tau|/2}.$$

We choose

$$\mathbf{V}_1(t,x) = -\int_{-\infty}^{t} S_Q^\Lambda(t,\tau)\mathbf{r}_c(\tau)d\tau$$

$$\Rightarrow |\mathbf{V}_1(t)|_{E^k} \leqslant CA^{1/4}e^{\delta_0 ct}, \quad t \leqslant 0.$$

For the general case $\mathbf{V}_j$ we use **induction argument**.

Theorem:

For every $N \in \mathbb{N}$, *there exists*

$$\boldsymbol{U}_{ap} = \boldsymbol{Q}_c + \boldsymbol{V} = \boldsymbol{Q}_c + \sum_{j=1}^{N} \rho^j \boldsymbol{V}_j(t,x),$$

where $\boldsymbol{V}_j \in C_b^\infty(\mathbb{R}_-, H^\infty(\mathbb{R}))$ *such that*

$$|\boldsymbol{V}_j|_{E^s} \leqslant A^{(2j-1)/4} C_{s,j}(\delta_0) e^{-j\delta_0 c|t|} \quad \forall t \leqslant 0. \tag{2}$$

In addition, $\boldsymbol{U}_{ap}$ *is an approximate solution of (ZWW) in the sense that the remainder* $\boldsymbol{r}_{ap}$ *defined as* $\partial_t \boldsymbol{U}_{ap} - \mathcal{F}(\boldsymbol{U}_{ap}) = \boldsymbol{r}_{ap}$ *satisfies the exponential decay*

$$|\boldsymbol{r}_{ap}|_{E^s} \leqslant A^{(2N+1)/4} C_{N,s}(\delta_0) \rho^{N+1} e^{-(N+1)\delta_0 c|t|} \quad \forall t \leqslant 0.$$

We point out that $\rho = e^{-\delta_0 A}$, which means that $A^{(2N+1)/4}\rho^{N+1}$ shall not grow to infinity for a growing larger A.

Step 5: Construction of the exact solution

We need to find the exact solution $\mathbf{U} = \mathbf{U}_{ap} + \mathbf{U}_r$, where $\mathbf{U}_r$ needs to be the solution to

$$\partial_t \mathbf{U}_r = \mathcal{F}(\mathbf{U}_{ap} + \mathbf{U}_r) - \mathcal{F}(\mathbf{U}_{ap}) - \mathbf{r}_{ap}. \qquad (3)$$

Existence of global solution for (3)

Proposition:

Let $p \geqslant 2$. For N large enough and ρ sufficiently small (A sufficiently large) in the definition of $\boldsymbol{V}$, there exists a solution $\boldsymbol{U}_r = (\eta_r, \varphi_r)^t \in L^\infty\left((-\infty, 0], H^{m+4} \times H^{7/2}\right)$ to

$$\begin{cases} \partial_t \boldsymbol{U}_r = \mathcal{F}(\boldsymbol{U}_{ap} + \boldsymbol{U}_r) - \mathcal{F}(\boldsymbol{U}_{ap}) - \boldsymbol{r}_{ap}, \\ \boldsymbol{U}_r(0) \text{ fixed}, \end{cases} \qquad (4)$$

such that $h\|a_\varepsilon\|_{L^\infty} - \|\eta_{ap}\|_{L^\infty} - \|\eta_r\|_{L^\infty} \geqslant h_{min} > 0$, and

$$|\boldsymbol{U}_r|_{X^{m+4} \times X^{m+7/2}} \leqslant A^{(2N-1)/4} \rho^{N+1} e^{-N\delta_0 c|t|} \quad \forall t \leqslant 0. \qquad (5)$$

Step 6: Proving the solution is a soliton-like solution.

We are left to prove

$$\lim_{t \to -\infty} |\mathbf{U}(t) - \mathbf{R}(t)|_{H^s} = 0. \tag{6}$$

From the definition of $\mathbf{U}$,

$$\mathbf{U} = \mathbf{R} + \sum_{j=1}^{N} \rho^j \mathbf{V}_j + \mathbf{U}_r.$$

The terms $\mathbf{V}_j$ and $\mathbf{U}_r$ satisfy a decay estimation each (deduced from (2) and (5)) for every $t \leqslant 0$:

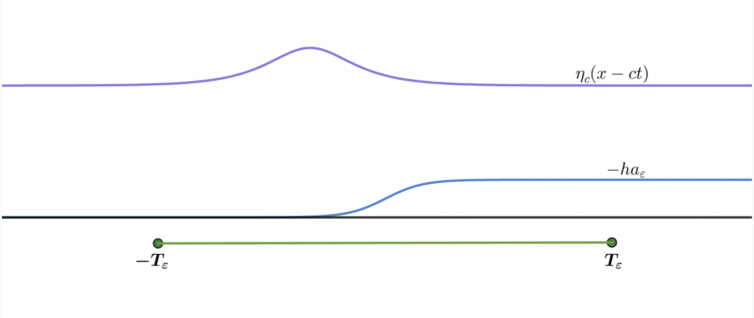
$$|\mathbf{V}_l|_{H^s} \leqslant A^{(2l-1)/4} C_{s,l}(\delta_0) e^{-l\delta_0 c|t|} \quad \text{and} \quad |\mathbf{U}_r|_{H^s} \leqslant A^{(2N-1)/4} \rho^{N+1} e^{-N\delta_0 c|t|}.$$

Consequently, we conclude (6).

Current work

The interaction regime.

We can define the interaction regime as $[-T_\varepsilon, T_\varepsilon]$, for $T_\varepsilon = \varepsilon^{-1-\alpha}$, $\alpha > 0$ small.



The natural next questions:

What happens in the interaction regime? Moreover, how does the bottom influence what comes out of the interaction regime?

Thank you!