Global Existence and Long Time Behavior in the 1+1 dimensional Principal Chiral Model with Applications to Solitons

New trends in Mathematics of Dispersive, Integrable and Nonintegrable Models in Fluids, Waves and Quantum Physics. BIRS

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Outline

1. Introduction

2. Main Results: Principal Chiral Field Equation
   - Local Existence
   - Global Existence
   - Long Time Behavior
   - Aplications to Solitons Solutions

General Relativity: Elements of the Lorentzian geometry

- A spacetime is a time-oriented (3+1)-dimensional Lorentzian manifold $(\mathcal{M}, \tilde{g})$, $\tilde{g}$ is a Lorentzian metric with signature (-,+,+,+).

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Mathematical general relativity

- Previous result
  - Well posed Cauchy problem, discovery by Choquet-Bruhat, 1952.
  - The singularity theorems of Penrose and Hawking, 1965.
  - Global aspects of the Cauchy problem, by Choquet-Bruhat and Geroch, 1969.
  - Inverse scattering transform for the Einstein equation, by Belinski and Zakharov, 1978.
  - Static and stationary multiple soliton solutions to the Einstein equations, P. Letelier, 1985.
  - Soliton solutions to the vacuum Einstein equations obtained from a nondiagonal seed solution, P. Letelier, 1986.
  - Prove the stability of the Kerr solution, Bernard Whiting,1989.
  - The global nonlinear stability of the Minkowski spacetime, D. Christodoulou and S. Klainerman, 1993.

Mathematical general relativity

- Current results
  - General definition of "conserved quantities" in general relativity and other theories of gravity, R. Wald, 2010.
  - Global well-posedness for a model for Einstein equations with additional compact dimensions, C. Huneau and A. Stingo, 2021.
  - Impulsive gravitational wave interaction, J. Luk and M. Van de Moortel, 2021.
  - Kerr stability for small angular momentum, S. Klainerman and J. Szeftel, 2021.
  - The non-linear stability of the Schwarzschild family of black holes, Gustav Holzegel , 2020.
  - More recently, work by Mihalis Dafermos, Igor Rodnianski, and collaborators considerably strengthened the results by Kay and Wald by proving decay of solutions to the scalar wave equation for the more general case of a Kerr black-hole background.

Belinski-Zakharov spacetimes

The Setup

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The spacetime metric in matrix form:

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

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Then the spacetime interval[2] is a simplified block diagonal form:

$$ds^2 = f(t,x)(dx^2 - dt^2) + g_{ab}(t,x)dx^a dx^b, \quad x^a = \{y, z\}, \quad e = 1 \quad (2)$$

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Includes: Kasner metric, Kerr metric, Kerr-Nut metric, Einstein-Rosen metric, Schwarzschild metric, Bianchi models, and others.

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Einstein's Soliton solutions

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$$ds^2 = f(t,r)(-dt^2 + dr^2) + e^{\Lambda(t,r)}(rd\phi)^2 + e^{-\Lambda(t,r)}dz^2$$

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Schwarzschild metric: **Kruskal-Szekeres coordinates**

$$ds^2 = -\frac{4r_s^2}{r}e^{-r/r_s}(dT^2 - dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

with $T^2 - R^2 = (1 - r/r_s)e^{r/r_s}$.

$$R_{\mu\nu}(\widetilde{g}) = 0.$$

The first one follows from equations $R_{ab} = 0$, this equation can be written as single matrix equation

$$\partial_t \left(\alpha \partial_t g g^{-1}\right) - \partial_x \left(\alpha \partial_x g g^{-1}\right) = 0, \quad \det g = \alpha^2. \tag{4}$$

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$$\partial_t^2 \alpha - \partial_x^2 \alpha = 0. \tag{5}$$

This is the so-called *trace* equation; the function $\alpha(t, x)$ satisfies the 1D wave equation.

Gravisolitons

- **Inverse Scattering Transfrom.**

The term gravi-soliton refers to the explicit solutions generated by the Belinski-Zakharov transform.

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In 2016, the first detection of gravitational waves by the twin LIGO[3] readers, produced by the merger of two black holes, was announced.

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New Coodenates

One writes $g = RDR^T$, where D is a diagonal matrix and R is a rotation matrix, of the form[4].

$$D = \begin{pmatrix} \alpha e^{\Lambda} & 0 \\ 0 & \alpha e^{-\Lambda} \end{pmatrix}, \quad R = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{6}$$

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$$g = \alpha \begin{pmatrix} \cosh\Lambda + \cos 2\phi \sinh\Lambda & \sin 2\phi \sinh\Lambda \\ \sin 2\phi \sinh\Lambda & \cosh\Lambda - \cos 2\phi \sinh\Lambda \end{pmatrix}.$$

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Now, with this representation, the equation (1) read

$$\begin{cases} \boxed{\partial_t \left(\alpha \partial_t g g^{-1}\right) - \partial_x \left(\alpha \partial_x g g^{-1}\right) = 0,} \\ \partial_t(\alpha \partial_t \Lambda) - \partial_x(\alpha \partial_x \Lambda) = 2\alpha \sinh 2\Lambda((\partial_t \phi)^2 - (\partial_x \phi)^2), \\ \partial_t(\alpha \sinh^2 \Lambda \partial_t \phi) - \partial_x(\alpha \sinh^2 \Lambda \partial_x \phi) = 0, \\ \partial_t^2 \alpha - \partial_x^2 \alpha = 0, \end{cases} \tag{7}$$

and

$$\partial_t^2(\ln f) - \partial_x^2(\ln f) = G, \tag{8}$$

where $G = G[\Lambda, \phi, \alpha]$ is given by

$$\begin{aligned} G := & -\left(\partial_t^2(\ln \alpha) - \partial_x^2(\ln \alpha)\right) - \frac{1}{2\alpha^2}((\partial_t \alpha)^2 - (\partial_x \alpha)^2) \\ & - \frac{1}{2}((\partial_t \Lambda)^2 - (\partial_x \Lambda)^2) - 2\sinh^2 \Lambda((\partial_t \phi)^2 - (\partial_x \phi)^2). \end{aligned} \tag{9}$$

Principal Chiral Field Equation

$$\partial_t \left(\partial_t g g^{-1}\right) - \partial_x \left(\partial_x g g^{-1}\right) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{10}$$

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The system (11) is a Hamiltonian system, having the conserved energy

$$E[\Lambda, \phi](t) := \int \left(\frac{1}{2}((\partial_x \Lambda)^2 + (\partial_t \Lambda)^2) + 2 \sinh^2 \Lambda((\partial_x \phi)^2 + (\partial_t \phi)^2) \right) dx.$$

Y. Hadad, *Integrable Nonlinear Relativistic Equations.*

R. Wald and A. Zoupas, *General definition of "conserved quantities" in general relativity and other theories of gravity.*

Classical Local Existence

Let us write the function $\Lambda(t, x)$ in the form

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$$\begin{cases} \Psi = [\tilde{\Lambda}, \phi], \quad \partial\Psi = [\partial_t\tilde{\Lambda}, \partial_x\tilde{\Lambda}, \partial_t\phi, \partial_x\phi], \quad F(\Psi, \partial\Psi) = [F_1, F_2], \\ |\partial\Psi|^2 = |\partial_t\tilde{\Lambda}|^2 + |\partial_x\tilde{\Lambda}|^2 + |\partial_t\phi|^2 + |\partial_x\phi|^2, \\ F_1(\Psi, \partial\Psi) := 2\sinh(2\lambda + 2\tilde{\Lambda})((\partial_x\phi)^2 - (\partial_t\phi)^2), \\ F_2(\Psi, \partial\Psi) := \dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})}(\partial_t\phi\partial_t\tilde{\Lambda} - \partial_x\phi\partial_x\tilde{\Lambda}). \end{cases}$$

$$\begin{cases} \partial_\alpha(m^{\alpha\beta}\partial_\beta \Psi) = F(\Psi, \partial\Psi) \\ (\Psi, \partial_t \Psi)|_{\{t=0\}} = (\Psi_0, \Psi_1) \in \mathcal{H}. \end{cases} \tag{13}$$

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Where $m^{\alpha\beta}$ are the components of the Minkowski metric with $\alpha, \beta \in \{0, 1\}$, and

$$(\Psi, \partial_t\Psi) \in \mathcal{H} := H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

We are also going to impose the following condition on the initial data

$$||(\Psi_0, \Psi_1)||_{\mathcal{H}} \leq \frac{\lambda}{2D}. \tag{14}$$

Proposition 1.

If (Ψ_0, Ψ_1) satisfies the condition (14) with an appropriate constant D, then:

(1). (Existence and uniqueness of local-in-time solutions). There exists T (depende of the initial data and λ) such that there exists a (classical) solution Ψ to (12) with

$$(\Psi, \partial_t \Psi) \in L^\infty([0, T]; \mathcal{H}).$$

Moreover, the solution is unique in this function space.

(2). (Continuous dependence on initial data). Let Ψ_0^i, Ψ_1^i be sequence such that $\Psi_0^i \longrightarrow \Psi_0$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ and $\Psi_1^i \longrightarrow \Psi_1$ in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ as $i \longrightarrow \infty$. Then taking $T > 0$ sufficiently small, we have

$$||(\Psi^{(i)} - \Psi, \partial_t(\Psi^{(i)} - \Psi))||_{L^\infty([0,T];\mathcal{H})} \longrightarrow 0.$$

Here Ψ is the solution arising from data (Ψ_0, Ψ_1) and $\Psi^{(i)}$ is the solution arising from data $(\Psi_0^{(i)}, \Psi_1^{(i)})$.

Main idea of the proof: Use energy estimates for the wave equation and bootstrap method[5].

[5] Sogge Christopher. *Lectures on Non-Linear Wave Equations*

Global Existence

- Klainerman with the pioneering works *The null condition and global existence to nonlinear wave equations*, 1986 (in three space dimensions).
- and by Demetrios Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, 1986 (in three space dimensions).
- Serge Alinhac, *The null condition for quasilinear wave equations in two space dimensions*, 2001.

In one space dimension case waves:

- Luli, Yang and Yu in 2018, *On one-dimension semi-linear wave equations with null conditions.*
- Leonardo Abbrescia and Willie Yeung in 2020, *Geometric analysis of* $1+1$ *dimensional quasilinear wave equations.*
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The classical null form, can be introduced as the bilinear form given by

$$Q_0(\phi, \tilde{\Lambda}) = m^{\alpha\beta} \partial_\alpha \phi \partial_\beta \tilde{\Lambda}, \tag{15}$$

where $m_{\alpha\beta}$ to denote the standard Minkowski metric on $\mathbb{R}^{1+1}$.

$$(PCFE) \begin{cases} \partial_t^2 \tilde{\Lambda} - \partial_x^2 \tilde{\Lambda} = -2\sinh(2\lambda + 2\tilde{\Lambda})((\partial_x \phi)^2 - (\partial_t \phi)^2), \\ \partial_t^2 \phi - \partial_x^2 \phi = -\dfrac{\sinh(2\lambda + 2\tilde{\Lambda})}{\sinh^2(\lambda + \tilde{\Lambda})}(\partial_t \phi \partial_t \tilde{\Lambda} - \partial_x \phi \partial_x \tilde{\Lambda}). \end{cases}$$

Teorema 1.

There exits ε_0 sufficiently small such that if the size of the data at time zero $(\phi, \partial_t \phi, \tilde{\Lambda}, \partial_t \tilde{\Lambda})(0)$ is $\varepsilon < \varepsilon_0$, there is a solution that remains smooth for all time in (PCFE).

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$$S_{t_0} := \{(t,x) : \ t = t_0\}.$$

$$D_{t_o} := \{(t,x) : \ 0 \leq t \leq t_0\}, \quad D_{t_0} = \bigcup_{0 \leq t \leq t_0} S_{t_0}.$$

$$C_{t_0, \underline{u}_0} := \left\{(t,x) : \ u = \frac{t-x}{2} = \underline{u}_0, \, 0 \leq t \leq t_0 \right\}.$$

Sketch of the proof

- Consider the two null vector fields defined globally as

$$L = \partial_t + \partial_x, \quad \underline{L} = \partial_t - \partial_x,$$

then,

$$(\partial_x\phi)^2 - (\partial_t\phi)^2 = Q_0(\phi,\phi) = 2L\phi\underline{L}\phi, \tag{16}$$

$$\partial_x\phi\partial_x\tilde{\Lambda} - \partial_t\phi\partial_t\tilde{\Lambda} = Q_0(\phi,\tilde{\Lambda}) = \frac{1}{2}L\phi\underline{L}\tilde{\Lambda} + \frac{1}{2}L\tilde{\Lambda}\underline{L}\phi. \tag{17}$$

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- Can be proved in a simple way that the null structure is "preserved" after diferentiating:

$$\partial_x Q_0(\phi, \tilde{\Lambda}) = Q_0(\partial_x \phi, \tilde{\Lambda}) + Q_0(\phi, \partial_x \tilde{\Lambda}). \tag{18}$$

- Also, based on this, we have the following inequality

$$Q_0(\partial_x^p \phi, \partial_x^q \phi) \lesssim |L\partial_x^p \phi||\underline{L}\partial_x^q \phi| + |\underline{L}\partial_x^p \phi||L\partial_x^q \phi|. \tag{19}$$

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- We will use the bootstrap method
  We define the space-time weighted energy norms:

$$\mathcal{E}_k(t) = \int_{S_t} \left[(1+|\underline{u}|^2)^{1+\delta} |\underline{L}\partial_x^k \tilde{\Lambda}|^2 + (1+|u|^2)^{1+\delta} |L\partial_x^k \tilde{\Lambda}|^2 \right] dx,$$

$$\mathcal{F}_k(t) = \sup_{u\in\mathbb{R}} \int_{C_{t,u}} (1+|\underline{u}|^2)^{1+\delta} |\underline{L}\partial_x^k \tilde{\Lambda}|^2 d\tau$$
$$+ \sup_{\underline{u}\in\mathbb{R}} \int_{C_{t,\underline{u}}} (1+|u|^2)^{1+\delta} |L\partial_x^k \tilde{\Lambda}|^2 d\tau.$$

Long time behavior

Energy and momentum densities:

$$
\begin{aligned}
p(t,x) &:= \partial_x \Lambda \partial_t \Lambda + 4\sinh^2(\Lambda)\partial_x\phi\partial_t\phi, \\
e(t,x) &:= \frac{1}{2}((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_t\phi)^2).
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Lemma: Continuity equations

Using the definition above in Eq. (20), one has the following continuity equations

$$\partial_t p(t,x) = \partial_x e(t,x), \quad \partial_t e(t,x) = \partial_x p(t,x), \tag{21}$$

and the inequality

$$|p(t,x)| \leq e(t,x). \tag{22}$$

Lemma: Energy conservation

If $\Lambda(t,x), \phi(t,x)$ are the solutions of (10) with $\Lambda(t,x) \in C_0^\infty(\mathbb{R})$ and $\phi(x) \in C_0^\infty(\mathbb{R})$ then the energy of the system is conserved, that is

$$\frac{d}{dt} E[\Lambda, \phi](t) = 0.$$

Virial estimates

Considerations:

- In what follows, we consider $|t| \geq 2$ only, and

$$\omega(t) := \frac{t}{\log^2(t)}, \qquad \frac{\omega'(t)}{\omega(t)} = \frac{1}{t}\left(1 - \frac{2}{\log(t)}\right). \tag{23}$$

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- Let $\rho := \tanh(\cdot)$, $v \in (-1, 1)$, let $\mathcal{I}(t)$ be defined as[6]

$$\mathcal{I}(t) := -\int \rho\left(\frac{x - vt}{\omega(t)}\right)(\partial_x \Lambda \partial_t \Lambda + 4\partial_x \phi \partial_t \phi \sinh^2(\Lambda))dx \tag{24}$$

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Lema: Virial identity

We have

$$\frac{d}{dt}\mathcal{I}(t) = \frac{\omega'(t)}{\lambda(t)}\int \frac{x-vt}{\omega(t)}\rho'\left(\frac{x-vt}{\omega(t)}\right)\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right)$$
$$+\frac{1}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\left(\frac{1}{2}(\partial_x\Lambda)^2 + 2(\partial_t\phi)^2\sinh^2(\Lambda)\right)$$
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$$+\frac{v}{\omega(t)}\int \rho'\left(\frac{x-vt}{\omega(t)}\right)\left(\partial_x\Lambda\partial_t\Lambda + 4\partial_x\phi\partial_t\phi\sinh^2(\Lambda)\right) \tag{25}$$

Lema 4.

Let $\omega(t)$ given as in (23). Assume that the solution $(\Lambda, \phi)(t)$ of the system (PCFE) satisfies

$$E[\Lambda, \phi](t) < +\infty$$

then we have the averaged estimate

$$\int_2^{\infty} \frac{1}{\omega(t)} \int \operatorname{sech}^2\left(\frac{x - vt}{\omega(t)}\right) e(t,x)dxdt \lesssim 1, \tag{26}$$

Moreover, there exists an increasing sequence of times $t_n \longrightarrow +\infty$ such that

$$\lim_{n \longrightarrow +\infty} \int \operatorname{sech}^2\left(\frac{x - vt}{\omega(t_n)}\right) e(t_n, x)dx = 0. \tag{27}$$

Integration of the dynamics

Teorema 2.

Let $(\Lambda, \Lambda_t, \phi, \phi_t)$ be a global solution to (PCFE) such that its energy is conserved and finite. Then, for any $v \in (-1, 1)$ and $\omega(t) = t^2 \log^{-1} t$, one has

$$\lim_{t\to+\infty} \int_{vt-\omega(t)}^{vt+\omega(t)} \left((\partial_x\Lambda)^2 + (\partial_t\Lambda)^2 + \sinh^2\Lambda((\partial_x\phi)^2 + (\partial_t\phi)^2)\right)(t,x)dx = 0.$$

Aplications to Solitons Solutions

- Belinskii and Zakharov in Relativistically invariant two dimensional models of field theory integrable by inverse scattering problem method proposed that the Eqn. (4) has $N-$soliton solutions.

Aplications to Solitons Solutions

- Belinskii and Zakharov in Relativistically invariant two dimensional models of field theory integrable by inverse scattering problem method proposed that the Eqn. (4) has $N-$soliton solutions.
- Hadad in Integrable Nonlinear Relativistic Equations also showed explicit examples of soliton solutions for the equation (PCFE) using diagonal backgrounds.

$$g^{(0)} = \begin{bmatrix} e^{\Lambda^{(0)}} & 0 \\ 0 & e^{-\Lambda^{(0)}} \end{bmatrix}$$

The function $\Lambda^{(0)}(t,x)$ satisfies

$$\partial_x^2 \Lambda^{(0)} - \partial_t^2 \Lambda^{(0)} = 0.$$

Singular Soliton

One-soliton solution, with $\Lambda^{(0)} = t$ (time-like) and $\phi^{(0)} = 0$. With a fixed parameter $\mu > 1$, one has

$$Q_c(\cdot) = \sqrt{c}\,\mathrm{sech}(\sqrt{c}(\cdot)),$$

$$c = \left(\frac{2\mu}{\mu^2 - 1}\right)^2, \quad v = -\frac{\mu^2 + 1}{2\mu} < -1, \quad \text{and} \quad x_0 = \frac{\ln|\mu|}{\sqrt{c}}.$$

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Traveling superluminal soliton which travels to the left:

$$g^{(1)} = \begin{bmatrix} \dfrac{e^t Q_c(x - vt)}{Q_c(x - vt - x_0)} & -\dfrac{1}{c} Q_c(x - vt) \\ -\dfrac{1}{c} Q_c(x - vt) & \dfrac{e^{-t} Q_c(x - vt)}{Q_c(x - vt + x_0)} \end{bmatrix}$$

Yaron Hadad.

Lemma

One has,

$$\Lambda(t,x) = \ln(|v|\cosh(t)) + \ln\left(1 - \frac{\tanh(t)\tanh(\gamma)}{|v|\sqrt{c}} + \sqrt{\left(1 - \frac{\tanh(t)\tanh(\gamma)}{|v|\sqrt{c}}\right)^2 - \frac{\mathrm{sech}^2(t)}{|v|^2}}\right),$$

$$\phi(t,x) = \frac{\pi}{4} - \frac{1}{2}\arctan\left(\cosh(t)\cosh(\sqrt{c}(x-vt))(\tanh(\sqrt{c}(x-vt)) + v\sqrt{c}\tanh(t)\right)$$

with $\gamma := \sqrt{c}(x-vt)$. For E_{mod} the previous solution gives

$$E_{\text{mod}}[\Lambda,\phi](t) = 0.$$

where

$$E_{\text{mod}}[\Lambda,\phi](t) := \int\left(\frac{1}{2}(((\partial_t\Lambda)^2 - 1) + (\partial_x\Lambda)^2) + 2\sinh^2(\Lambda)((\partial_x\phi)^2 + (\partial_x\phi)^2)\right).$$

Finite energy solitons

Considerations:

1. Take a function $\theta \in C_c^\infty(\mathbb{R})$.
2. Consider the constraint $0 < \mu < 1$.
3. For any $\lambda > 0$, and $\varepsilon > 0$ small, let

$$\Lambda_\varepsilon^{(0)} := \lambda + \varepsilon\theta(t + x), \quad \phi^{(0)} := 0.$$

4. Finite Energy

$$E[\Lambda_\varepsilon^{(0)}, \Lambda_{\varepsilon,t}^{(0)}, \phi^{(0)}, \phi_t^{(0)}] < +\infty.$$

The corresponding 1-soliton is now

$$g^{(1)} = \begin{bmatrix} \dfrac{e^{\lambda+\varepsilon\theta}\,\mathrm{sech}(\beta(\lambda+\varepsilon\theta))}{\mathrm{sech}(\beta(\lambda+\varepsilon\theta)-x_0)} & -\dfrac{1}{\sqrt{c}}\,\mathrm{sech}(\beta(\lambda+\varepsilon\theta)) \\ -\dfrac{1}{\sqrt{c}}\,\mathrm{sech}(\beta(\lambda+\varepsilon\theta)) & \dfrac{e^{-(\lambda+\varepsilon\theta)}\,\mathrm{sech}(\beta(\lambda+\varepsilon\theta))}{\mathrm{sech}(\beta(\lambda+\varepsilon\theta)+x_0)} \end{bmatrix}, \quad (28)$$

with $\beta = \frac{\mu+1}{\mu-1}$.

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Corollary

Suitable perturbations of any soliton as in (28), whit the conditions (1)-(3), are globally well-defined.

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Corollary

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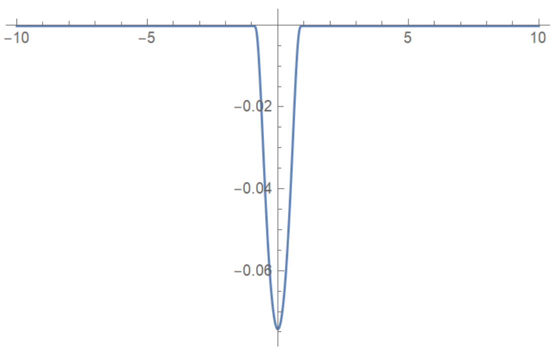
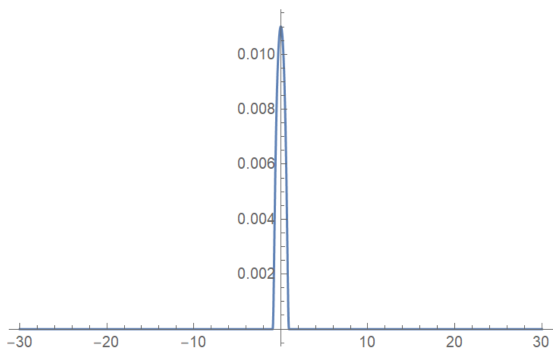
Example

Let us choose

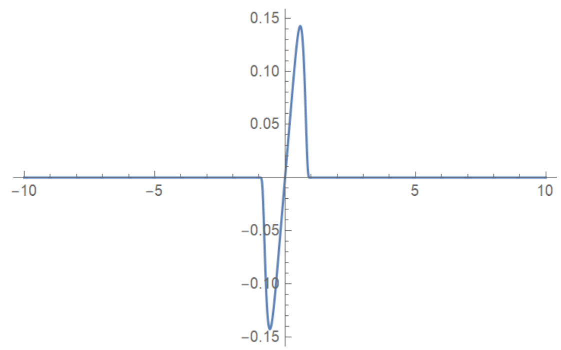
$$\theta(x) := \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$



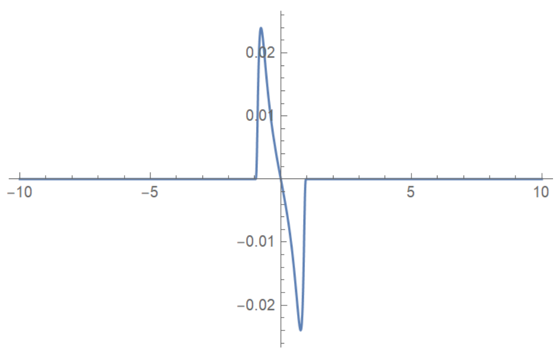
(a) $\tilde{\Lambda}(t = 0, x)$



(b) $\phi(t = 0, x)$



(c) $\partial_t \tilde{\Lambda}|_{\{t=0\}}$



(d) $\partial_t \phi|_{\{t=0\}}$

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Thanks!