Asymptotic stability manifolds for solitons in the generalized Good Boussinesq equation

New trends in Mathematics of Dispersive, Integrable and Nonintegrable Models in Fluids, Waves and Quantum Physics, BIRS

Christopher Maulén
Universität Bielefeld, CRC 1283

October 10, 2022

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Boussinesq Models

In the 1870's, J. Boussinesq deduced a system of equations to **describe two-dimensional irrotational** and **inviscid fluids** in a **uniform rectangular channel** with **flat bottom**. He was the first to give a favorable explanation to the traveling-waves, solitons, or solitary waves solutions discovered by Scott Russell (1844) thirty years earlier, which remained in their form and travelled with constant velocity.

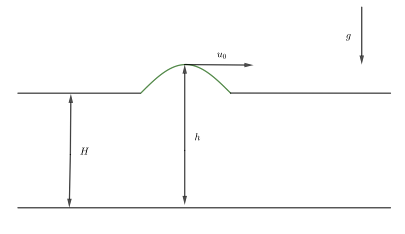
He made an approximation of the Eulerian problem to describe the two-way propagation of small amplitude gravity waves on the surface of the water in a canal, and obtained the following equation:

$$\frac{d^2}{dt^2}h = gH\frac{d^2}{dx^2}h + gH\frac{d^2}{dx^2}\left(\frac{3h^2}{2H} + \frac{H^2}{3}\frac{d^2}{dx^2}h\right),$$

for describing a two-dimensional flow of shallow-water waves having small amplitudes.

Here

- h is the height of the fluid
- u_0 is its speed.
- g is the gravity constant.
- H the depth of the liquid without perturbations (constant).



Bad and Good-Boussinesq

The above equation is called **Bad Boussinesq**, which simplifies to the adimensional model

$$\partial_t^2 u - \partial_x^4 u - \partial_x^2 u - \partial_x^2(u^2) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \qquad \text{(BadB)}$$

But this equation is strongly (linearly) ill-posed.

This bad behavior is not present when the plus sign is considered in the approximation, obtaining

$$\partial_t^2 \phi + \partial_x^4 \phi - \partial_x^2 \phi - \partial_x^2(\phi^2) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \qquad \text{(GoodB)}$$

which is called **Good–Boussinesq**.

Results for Good Boussinesq equation

Recall that the Good Boussinesq model, in its simplified form, is given by:

$$\partial_t^2\phi + \partial_x^4\phi - \partial_x^2\phi + \partial_x^2(f(\phi)) = 0,$$

and if formally $u = \phi$ and $v = \partial_x^{-1}\partial_t\phi$, has the following representation as 2×2 system:

$$\begin{cases} \partial_t u = \partial_x v \\ \partial_t v = \partial_x(-\partial_x^2 u + u - f(u)), \end{cases} \qquad (g\text{GB})$$

which is Hamiltonian, and has the following associated conserved quantities:

$$E[u,v] = \frac{1}{2}\int \left[v^2 + u^2 + (\partial_x u)^2 - 2F(u)\right]dx \qquad \text{(Energy)},$$

$$P[u,v] = \int uv dx \qquad \text{(Momentum)}.$$

These laws define a standard **energy space** $(u,v) \in H^1 \times L^2$.

Solitary Waves

A solitary wave is a solution to (gGB) of the form

$$(u,v) = (Q_c, -cQ_c)(x - ct - x_0), \quad |c| < 1, \quad x_0 \in \mathbb{R},$$

with Q_c solving $(c^2-1)Q_c + Q_c'' + f(Q_c) = 0$ in $H^1(\mathbb{R})$.

When the nonlinearity has the form $f(s) = |s|^{p-1}s$ for $p > 1$, standing solitary waves have the form

$$u(t,x) = Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)}\right)^{1/(p-1)}, \quad v(t,x) = 0. \tag{1}$$

Previous Results

The fundamental works

- LWP and GWP for small data (Bona and Sachs 1988)
- Existence of solitary waves for velocities $c^2 < 1$ (Bona and Sachs 1988)
- GWP in the energy space in the case of small data (Linares 1993,2005).
- For $p = 2$, GWP in $H^s(\mathbb{R})$, for $s \geq -1/2$, and IP for $s < -1/2$. (Kishimoto 2013)

We are motivated by the long time behavior problem for solitary waves of the gGB in the case where $f(s) = |s|^{p-1}s$ for $p > 1$.

- **Solitary waves are stable** if the speed c obeys the condition $(p-1)/4 < c^2 < 1$ and $p > 4$ (Bona and Sachs 1988).
- Solutions with **initial data arbitrarily near the ground state** $(c = 0)$ that **blow-up** in finite time (Liu 1995).

First Main Theorem

Theorem (C. M (2021))

Let $p \geq 2$. There exists $\delta > 0$ such that if a global even-odd solution $(\phi, \partial_t \partial_x^{-1} \phi)$ of (gGB) satisfies for all $t \geq 0$,

$$\|(\phi, \partial_t \partial_x^{-1} \phi)(t) - (Q, 0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} < \delta,$$

then, for any $\gamma > 0$ small enough and any compact interval I of $\mathbb{R}$,

$$\lim_{t \to +\infty} \left(\|\phi(t) - Q\|_{L^2(I) \cap L^\infty(I)} + \|(1 - \gamma \partial_x^2)^{-1} \partial_t \phi(t)\|_{L^2(I)} \right) = 0.$$

Linearized system around the standing wave

Let us consider a perturbation in (gGB) of Q of the form

$$u(t,x) = Q(x) + w(t,x), \quad v(t,x) = z(t,x).$$

Then one can see that this perturbation satisfies the following linear system at first order:

$$\begin{cases} \partial_t w = \partial_x z \\ \partial_t z = \partial_x \mathcal{L} w, \end{cases} \iff \partial_t^2 w = \partial_x^2 \mathcal{L} w. \tag{2}$$

where

$$\mathcal{L}(w) = -\partial_x^2 w + V_0(x) w, \quad \text{with} \quad V_0(x) = 1 - f'(Q). \tag{3}$$

$\mathcal{L}$ is the classical Schrodinger operator associated to the soliton Q. This operator has been extensively studied by Chang-Gustafson-Nakanishi-Tsai (2007) for instance.

Linearized opertaor $-\partial_x^2\mathcal{L}$

We will assume the following: for any $p > 1$, the linear operator

$$-\partial_x^2\mathcal{L}(u) = \partial_x^4 u - \partial_x^2 u + \partial_x^2(pQ^{p-1}u), \tag{4}$$

has a unique eigenfunction $\phi_0(x)$ associated to a negative first eigenvalue $-\nu_0^2 < 0$, satisfying

$$-\partial_x^2\mathcal{L}(\phi_0) = -\nu_0^2\phi_0, \quad \langle \partial_x^{-1}\phi_0, \partial_x^{-1}\phi_0\rangle = 1, \quad |\phi_0(x)| \lesssim e^{-1^-|x|}. \tag{5}$$

Note that we also have $\partial_x^{-1}\phi_0$ well-defined, exponentially decreasing and part of L^2. Here $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{R})$, and 1^- is a number slightly below 1.

Linearized opertaor $-\partial_x^2\mathcal{L}$

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Note that we also have $\partial_x^{-1}\phi_0$ well-defined, exponentially decreasing and part of L^2. Here $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{R})$, and 1^- is a number slightly below 1.

The second eigenvalue of $-\partial_x^2\mathcal{L}$ is 0 but it is also a resonance in the classical sense (in $L^\infty\backslash L^2$), but the unique L^2 eigenvalue is $\phi_1(x) = c_1 Q'(x)$.

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Therefore, by the Spectral Theorem, orthogonal to ϕ_0 the operator $-\partial_x^2\mathcal{L}$ is nonnegative.

Let

$$\boldsymbol{Y}_{\pm} = \begin{pmatrix} \phi_0 \\ \pm\nu_0 \partial_x^{-1} \phi_0 \end{pmatrix}, \quad \boldsymbol{Z}_{\pm} = \begin{pmatrix} \partial_x^{-2} \phi_0 \\ \pm\nu_0^{-1} \partial_x^{-1} \phi_0 \end{pmatrix}. \tag{6}$$

These are even-odd functions, i.e. the first coordinate is even and the second odd.

The functions

$$\mathbf{u}_{\pm}(t,x) = e^{\pm\nu_0 t} \mathbf{Y}_{\pm}(x)$$

are solutions of the linearized problem (2), showing the presence of exponentially stable and unstable linear manifolds relevant for the dynamics of nonlinear solutions in a neighborhood of the soliton.

Theorem (C. M (2021))

Let $p \geq 2$, and

$$\mathcal{A}_0 = \left\{ \boldsymbol{\epsilon} \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \middle| \ \boldsymbol{\epsilon} \text{ is even-odd }, \|\boldsymbol{\epsilon}\|_{H^1 \times L^2} < \delta_0 \text{ and } \langle \boldsymbol{\epsilon}, \boldsymbol{Z}_+ \rangle = 0 \right\}.$$

There exist $C, \delta_0 > 0$, and a Lipschitz function $h : \mathcal{A}_0 \to \mathbb{R}$ with $h(0) = 0$ and $|h(\epsilon)| \leq C\|\epsilon\|^{3/2}_{H^1 \times L^2}$ such that, denoting

$$\mathcal{M} = \{(Q, 0) + \epsilon + h(\epsilon)Y_+ \text{ with } \epsilon \in \mathcal{A}_0\},$$

the following holds:

1. *If $\phi_0 \in \mathcal{M}$ then the solution of (gGB) with initial data ϕ_0 is global and satisfies, for all $t \geq 0$,*

$$\|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} \leq C\|\boldsymbol{\phi}_0 - (Q, 0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})}.$$

2. *If a global even-odd solution ϕ of (gGB) satisfies, for all $t \geq 0$,*

$$\|\boldsymbol{\phi}(t) - (Q, 0)\|_{H^1(\mathbb{R}) \times L^2(\mathbb{R})} \leq \frac{\delta_0}{2},$$

then for all $t \geq 0$, $\phi(t) \in \mathcal{M}$.

Idea of proof

We will consider $(u_1, u_2) \in H^1 \times L^2$ be an **even-odd perturbation** of the solitary waves, which are in some sense **orthogonal** to $\boldsymbol{Y}_+$ and $\boldsymbol{Y}_-$, and the flow on these directions: for a_1, a_2 unique,

$$\begin{cases} u(t,x) = Q(x) + a_1(t)\phi_0(x) + u_1(t,x), \\ v(t,x) = a_2(t)\nu_0\partial_x^{-1}\phi_0(x) + u_2(t,x). \end{cases}$$

where (see (5))

$$a_1(t) = \langle u(t) - Q, \partial_x^{-2}\phi_0\rangle, \quad a_2(t) = \frac{1}{\nu_0}\langle \partial_x v, \partial_x^{-1}\phi_0\rangle,$$

such that

$$\langle u_1(t), \partial_x^{-2}\phi_0\rangle = 0 = \langle u_2(t), \partial_x^{-1}\phi_0\rangle. \tag{7}$$

We have for all $t \in \mathbb{R}_+$

$$\|u_1(t)\|_{H^1} + \|u_2(t)\|_{L^2} + |a_1(t)| + |a_2(t)| \leq C_0\delta. \tag{8}$$

Moreover, using (5) and (7), (a_1, a_2) satisfies the following differential system

$$\begin{cases} \dot{a}_1 = \nu_0 a_2 \\ \dot{a}_2 = \nu_0 a_1 + \dfrac{N_0}{\nu_0}, \end{cases} \tag{9}$$

where

$$\begin{aligned} N &= \partial_x \left(f(Q) + f'(Q)(a_1\phi_0 + u_1) - f(Q + a_1\phi_0 + u_1) \right), \\ N^{\perp} &= \ N - N_0 \partial_x^{-1}\phi_0, \quad \text{and} \quad N_0 = \langle N, \partial_x^{-1}\phi_0 \rangle. \end{aligned} \tag{10}$$

Then, (u_1, u_2) satisfies the system

$$\begin{cases} \dot{u}_1 = \partial_x u_2 \\ \dot{u}_2 = \partial_x \mathcal{L}(u_1) + N^{\perp}, \end{cases} \tag{11}$$

with u_1 even and u_2 odd.

First Step

$$\begin{tikzcd}
(u_1, u_2) \arrow[dr] & \\
& (w_1, w_2)
\end{tikzcd}$$

Here

- (w_1, w_2) is localized version of (u_1, u_2) at A scale.
- $(u_1, u_2) \in H^1 \times L^2$ and an adequate weight function φ_A placed at scale A large.

$$\begin{cases} \dot{u}_1 = \partial_x u_2 \\ \dot{u}_2 = \partial_x \mathcal{L}(u_1) + N^{\perp}, \end{cases}$$

$$\begin{aligned} \frac{d}{dt}\int \varphi_A(x) u_1 u_2 \leq & -\frac{1}{2}\int \left[w_2^2 + 2(\partial_x w_1)^2 + \left(1 - C_1 A^{-1}\right) w_1^2\right] \\ & + C_1 a_1^4 + C_1 \int \operatorname{sech}(x)\, u_1^2, \end{aligned} \tag{12}$$

Second Step

$$
\begin{tikzcd}
& (u_1,u_2) \arrow[dl, "(\Lambda_\gamma^{-1}\mathfrak{L},\Lambda_\gamma^{-1})"'] \arrow[dr] & \\
(v_1,v_2) \arrow[dr] & & (w_1,w_2) \\
& (z_1,z_2) &
\end{tikzcd}
$$

Here

- (w_1, w_2) is localized version of (u_1, u_2) at A scale.
- (z_1, z_2) is localized version of (v_1, v_2) at B scale
- $\Lambda_\gamma^{-1} = (1 - \gamma\partial_x^2)^{-1}$
- $(v_1, v_2) \in H^1 \times H^2$, for an adequate weight function $\psi_{A,B}$, $B \ll A$.

$$
\begin{cases}
\dot{v}_1 = \mathcal{L}(\partial_x v_2) + G(x), \\
\dot{v}_2 = \partial_x v_1 + H(x),
\end{cases}
.
$$

$$
\begin{aligned}
\frac{d}{dt}\int \psi_{A,B} v_1 v_2 \leq & -\frac{1}{2}\int \left[z_1^2 + (V_0(x) - C_2 B^{-1})z_2^2 + 2(\partial_x z_2)^2\right] \\
& + B^{-1}C_2\left(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_2|a_1|^3,
\end{aligned}
\tag{13}
$$

Third Step

$$
\begin{tikzcd}
 & (u_1, u_2) \arrow[dl, "(\Lambda_\gamma^{-1}\mathfrak{L}, \Lambda_\gamma^{-1})"'] \arrow[dr] & \\
(v_1, v_2) \arrow[dr] & & (w_1, w_2) \\
 & (z_1, z_2) &
\end{tikzcd}
$$

Here

- (w_1, w_2) is localized version of (u_1, u_2) at A scale.
- (z_1, z_2) is localized version of (v_1, v_2) at B scale
- $\Lambda_\gamma^{-1} = (1 - \gamma\partial_x^2)^{-1}$

Following Kowalczyk-Martel-Muñoz, we have the following coercivity estimate in terms of the variables (w_1, w_2) and (z_1, z_2):

$$\int \operatorname{sech}(x) u_1^2 \lesssim B^{-1/2}\left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2\right) + B^{1/2}\|z_1\|_{L^2}^2 + B^{-4}\|\partial_x z_1\|_{L^2}^2 \tag{14}$$

Fourth Step

$$
\begin{tikzcd}
 & (u_1,u_2) \arrow[dl, "(\Lambda_\gamma^{-1}\mathcal{L},\Lambda_\gamma^{-1})"'] \arrow[dr] & \\
(v_1,v_2) \arrow[dr] \arrow[dd, "(\partial_x,\partial_x)"'] & & (w_1,w_2) \\
 & (z_1,z_2) & \\
(\tilde{v}_1,\tilde{v}_2) \arrow[ur] & &
\end{tikzcd}
$$

Here

- (w_1, w_2) is localized version of (u_1, u_2) at A scale.
- (z_1, z_2) is localized version of (v_1, v_2) at B scale
- $\Lambda_\gamma^{-1} = (1 - \gamma\partial_x^2)^{-1}$
- $(\tilde{v}_1, \tilde{v}_2) \in L^2 \times H^1$, for an adequate weight function $\psi_{A,B}$, $B \ll A$.

$$\begin{cases} \dot{\tilde{v}}_1 = \mathcal{L}(\partial_x \tilde{v}_2) + \tilde{G}(x), \\ \dot{\tilde{v}}_2 = \partial_x \tilde{v}_1 + \tilde{H}(x), \end{cases} .$$

$$\frac{d}{dt}\int \psi_{A,B}\tilde{v}_1\tilde{v}_2 \leq -\frac{1}{2}\int \left((\partial_x z_1)^2 + \left(V_0(x) - C_3 B^{-1}\right)(\partial_x z_2)^2 + 2(\partial_x^2 z_2)^2\right) + C_3|a_1|^3$$
$$+ C_3 B^{-1}\left(\|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2 + \|z_1\|_{L^2}^2 + B\|z_2\|_{L^2}^2\right) \tag{15}$$

Fifth Step

$$\begin{tikzcd}
& (u_1,u_2) \arrow[dl, "\left(\Lambda_\gamma^{-1}\mathfrak{L},\Lambda_\gamma^{-1}\right)"'] \arrow[dr] & \\
(v_1,v_2) \arrow[dr] \arrow[dd, "(\partial_x,\partial_x)"'] & & (w_1,w_2) \\
& (z_1,z_2) & \\
\left(\tilde{v}_1,\tilde{v}_2\right) \arrow[ur] & &
\end{tikzcd}$$

Here

- (w_1, w_2) is localized version of (u_1, u_2) at A scale.
- (z_1, z_2) is localized version of (v_1, v_2) at B scale
- $\Lambda_\gamma^{-1} = (1-\gamma\partial_x^2)^{-1}$

Finally, our last contribution is a transfer virial estimate that exchanges information between $\partial_x z_1$, $\partial_x z_2$ and $\partial_x^2 z_2$, in the form of

$$\begin{aligned}\frac{1}{2}\int(\partial_x z_1)^2 \leq \frac{d}{dt}\int \rho_{A,B}\tilde{v}_1 v_2 + C_4\int\left[(\partial_x^2 z_2)^2 + (\partial_x z_2)^2 + z_2^2 + z_1^2\right] \\ + C_4 B^{-3}\left(\|w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_4|a_1|^3.\end{aligned} \tag{16}$$

Final Step

Finally, we consider a functional $\mathcal{H}$ being a well-chosen linear combination of (12), (13), (15), (14) and (16). We get

$$\frac{d}{dt}\mathcal{H}(t) \leq -C_2 B^{-1}\left(\|w_1\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_2\|_{L^2}^2\right) + C_5|a_1|^3, \quad \text{for all } t \geq 0.$$

This final estimate allows us to close estimates, and prove local decay for u_1 after some standard change of variables from w_j to u_j.

Setting

$$b_+ = \frac{1}{2}(a_1 + a_2), \quad b_- = \frac{1}{2}(a_1 - a_2). \tag{17}$$

Recalling that (a_1, a_2) satisfies the following differential system

$$\begin{cases} \dot{a}_1 = \nu_0 a_2 \\ \dot{a}_2 = \nu_0 a_1 + \dfrac{N_0}{\nu_0}, \end{cases} \quad \text{or equivalently} \quad \begin{cases} \dot{b}_+ = \nu_0 b_+ + \dfrac{N_0}{2\nu_0} \\ \dot{b}_- = -\nu_0 b_- - \dfrac{N_0}{2\nu_0}. \end{cases} \tag{18}$$

We define now

$$\mathcal{B} = b_+^2 - b_-^2.$$

Combining the estimate of $\frac{d}{dt}\mathcal{H}$ and computing $\frac{d}{dt}\mathcal{B}$, it holds

$$\frac{d}{dt}\left(\mathcal{B} - 2B\frac{C_6}{C_2}\mathcal{H}\right) \geq \frac{\nu_0}{4}(a_1^2 + a_2^2) + C_6\left(\|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2\right). \tag{19}$$

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Estimate $|\mathcal{B}| \leq \delta^2$ is also clear from (8). Therefore, integrating estimates (19) on $[0, t]$ and passing the limit as $t \to \infty$, we have

$$\int_0^\infty \left[a_1^2 + a_2^2 + \|w_2\|_{L^2}^2 + \|\partial_x w_1\|_{L^2}^2 + \|w_1\|_{L^2}^2\right] dt \lesssim \delta.$$

We used a combination of virials to obtain the integrability in time of the $L^2 \times L^2$-norm of $(\phi(t) - Q, (1 - \gamma\partial_x^2)^{-1}\partial_t\phi(t))$, for any $\gamma > 0$ small enough, and in any compact interval I, i.e.,

$$\int_0^\infty \left(\|\phi(t) - Q\|_{L^2(I)}^2 + \|(1 - \gamma\partial_x^2)^{-1}\partial_t\phi(t)\|_{L^2(I)}^2\right) dt < \infty.$$

Thank you!!