$3d$ Navier-Stokes equations & the multifractal model

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Aim of this talk :

- What is the effect of blending the multifractal model (MFM) of Frisch & Parisi (1985) with the Navier-Stokes equations in a periodic box $[0,\ L]^3$

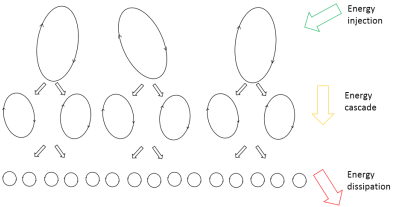
$$(\partial_t + \boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla P + \boldsymbol{f}(\boldsymbol{x}) \qquad \mathrm{div}\,\boldsymbol{u} = 0\,?$$

- Berengere Dubrulle & JDG : *A correspondence between the multifractal model of turbulence & the NSEs*, Phil. Trans. R. Soc. A **380**, 20210092.

Plan of this talk :

1. Summary of relevant results on the NS equations in both 3-dimensions and d-dimensions ($d = 2,\ 3$).
2. Summary of the MFM in its "Large Deviation Theory" format.
3. Lower bounds on the scaling parameter h & the multifractal spectrum $C(h)$ (co-dimension).
4. What are the consequences?

Turbulent cascades & higher derivatives in the NSEs



Standard "cartoon" of a turbulent cascade to small scales.

Define a doubly-labelled set of volume integrals for $1 \leq n < \infty\,;\, 1 \leq m \leq \infty$

$$H_{n,m,d} = \int_{V_d} |\nabla^n \boldsymbol{u}|^{2m} dV_d \qquad \text{in } d\text{-dimensions}$$

In dimensionless form :

$$F_{n,m,d} = \nu^{-1} L^{1/\alpha_{n,m,d}} H_{n,m,d}^{1/2m}\,, \qquad \alpha_{n,m,d} = \frac{2m}{2m(n+1)-d}\,,$$

1. Derivatives are sensitive to ever finer length scales in the flow.
2. Higher values of m pick out the larger spikes, with the $m = \infty$ case representing the maximum norm.

$\langle\cdot\rangle_T$ means time average up to time T : (JDG 2018, 2020 & based on FGT 1981)

Theorem

/ On periodic BCs with $n \geq 1$ & $1 \leq m \leq \infty$, d-dim NS-weak solutions obey ($d = 2, 3$)

$$\left\langle F_{n,m,d}^{(4-d)\alpha_{n,m,d}} \right\rangle_T \leq c_{n,m,d}\, Re^3 + O\left(T^{-1}\right).$$

- For $d = 3$ when $n = 1$, $m = 1$ gives the standard $\varepsilon \leq L^{-4}\nu^3 Re^3$ from which the Kolmogorov length λ_k is estimated

$$\lambda_k^{-1} = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \qquad \Rightarrow \qquad L\lambda_k^{-1} \leq Re^{3/4}.$$

- The above is a weak soln result : for full $d = 3$ regularity we would need

$$\left\langle F_{n,m,3}^{2\alpha_{n,m,3}} \right\rangle_T < \infty,$$

which is a result we **don't** have (JDG 2018).

Definition of a sequence of length scales $\lambda_{n,m,d}(t)$

Define a set of t-dependent length-scales $\{\lambda_{n,m,d}(t)\}$ s.t.

$$\left(\frac{L}{\lambda_{n,m,d}}\right)^{-d} H_{n,m,d} = \lambda_{n,m,d}^{-2m(n+1)+d}\nu^{2m}$$

from which we discover

$$\left(L\lambda_{n,m,d}^{-1}\right)^{n+1} = F_{n,m,d} \qquad \text{with} \qquad \alpha_{n,m,d} = \frac{2m}{2m(n+1)-d}$$

Result

For NS weak solutions, when $n \geq 1$ and $1 \leq m \leq \infty$

$$\left\langle L\lambda_{n,m,d}^{-1}\right\rangle_T \leq c_{n,m,d} Re^{\frac{3}{(4-d)(n+1)\alpha_{n,m,d}}} + O\left(T^{-1}\right).$$

The upper bound has a finite limit :

$$\lim_{\mathbf{n,m}\to\infty} \frac{\mathbf{3}}{(\mathbf{4-d})(\mathbf{n+1})\alpha_{\mathbf{n,m,d}}} = \frac{\mathbf{3}}{\mathbf{4-d}}$$

a result which has important consequences.

Scale invariance and K41

The Euler equations

$$(\partial_t + \boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \nabla P = 0 \qquad \text{div}\,\boldsymbol{u} = 0$$

have the scale invariance :

$$\boldsymbol{x}' = \lambda^{-1}\boldsymbol{x}\,, \qquad t' = \lambda^{h-1}t\,, \qquad \boldsymbol{u} = \lambda^{h}\boldsymbol{u}'$$

whereas the NS-equations are restricted to the value $h = -1$. All of the following can be found in Frisch (1995) or Benzi & Biferale (2008) :

• K41 suggests that, at a point $\boldsymbol{x}$ in a homogeneous, isotropic NS flow, the p-th order velocity structure function S_p should scale as

$$S_p(r) = \left\langle |\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})|^p \right\rangle_{st.av.} \sim r^{hp}\,.$$

• It also suggests that $h = \frac{1}{3}$ to ensure that the energy dissipation rate ε is homogeneous in space and time. Thus

$$S_p \sim r^{p/3}\,.$$

• When $p = 3$ the right hand side is equal to $-\frac{4}{5}\varepsilon r$ (the four-fifths law).

The Multifractal Model (MFM) of Frisch and Parisi : I

- Parisi and Frisch (1985) relaxed the enforcement of $h = \frac{1}{3}$ to allow a range values of h, provided the dissipation rate ε is constant "on the average".
- In the MFM's original formulation $P_r(h)$, the probability of observing a given scaling exponent h at the scale r was computed by assuming that each value of h belongs to a given fractal set of dimension $D(h)$.
- A more modern definition uses Large Deviation Theory where $P_r(h)$ is chosen as (see Eyink (2008) http://www.ams.jhu.edu/~eyink/Turbulence/notes/)

$$P_r(h) \sim r^{C(h)} \,.$$

$C(h)$ is the multi-fractal spectrum. It has encoded within it all the properties of flow intermittency. One can write $d = D(h) + C(h)$.

- The structure functions $S_p(r)$ are now expressed as

$$S_p(r) \sim r^{\zeta_p} \,, \qquad\qquad \zeta_p = \inf_h \left[hp + C(h)\right] \,.$$

A classic sign of intermittency is that ζ_p is a *concave curve below linear*.

- Paladin and Vulpiani (1987) suggested an h-dependent dissipation scale η_h

$$L\eta_h^{-1} \sim Re^{\frac{1}{1+h}} \,.$$

The NSEs and the MFM : I

We use the Paladin-Vulpiani scaling η_h to obtain the correspondence

$$H_{n,m} = L^{-3}\int_{\mathcal{V}_\ulcorner} |\nabla^n \boldsymbol{u}|^{2m}\, dV_d \quad \longleftrightarrow \quad \int_h \eta_h^{2m(h-n)} P_{\eta_h}(h)dh\,,$$

To pursue the idea proposed by Nelkin (1990) we use $\eta_h \sim \nu^{1+h}$

$$H_{n,m} \sim L^3 \nu^{\chi_{n,m}} \qquad\qquad \chi_{n,m} = \min_h \left(\frac{2m(h+1) + C(h) - 2m(n+1)}{1+h}\right)\,.$$

Use this in the LHS of Theorem 1 : i.e. the estimate for $\left\langle F_{n,m,d}^{(4-d)\alpha_{n,m,d}}\right\rangle_T$, and compare the result with the RHS in powers of ν ($\nu \to 0$) :

$$C(h) \geq 2m(n+1)\left(1 - \frac{3(1+h)}{4-d}\right) + \frac{3d(1+h)}{4-d}, \quad \forall (n,m) \geq 1\,.$$

In the limit $(n,m) \to \infty$ the RHS $\to \infty$ unless $h \geq (1-d)/3$.

Result

The only scaling exponents that have a nonzero probability are

$$h \geq h_{min} \qquad\qquad h_{min} = (1-d)/3\,.$$

When $d = 3$ we have the lower bound $h \geq -\frac{2}{3}$.

For $h \geq h_{min}$, the sharpest bound on $C(h)$, uniform in n, m, comes from $m = n = 1$

$$C(h) \geq 1 - 3h\,, \qquad \text{with} \qquad C(h_{min}) \geq d\,,$$

which is no better than the 4/5ths law. $C(h_{min}) \geq d$ is a feature allowed by Large Deviation Theory (Eyink) but has a low probability of occurrence.

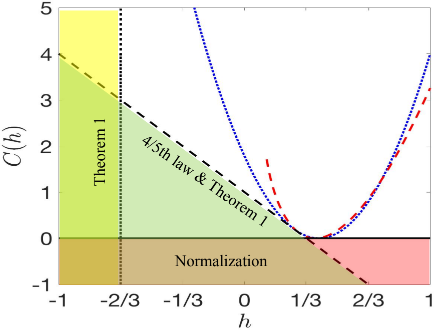


Figure: The admissibility range of $C(h)$ when $d = 3$ including $C(h) \geq 1 - 3h$. The blue dotted line : log-normal model with $b = 0.045$; red dashed line : log-Poisson model with $\beta = 2/3$.

Avoidance of the CKN singular set?

In $d = 3$ dimensions, the range of h is now

$$-2/3 \leq h \leq 1/3$$

thus implying a wide range of fractal dimensions.

1. Caffarelli, Kohn and Nirenberg (1982) developed the idea of suitable weak solutions of the $3d$ NSEs. The singular set in space-time has zero one-dimensional Hausdorff measure.

2. Their result shows that in the limit as solutions approach the CKN singular set, the velocity field $\boldsymbol{u}$ must obey

$$|\boldsymbol{u}| > \frac{const}{r}, \quad \text{as} \quad r \to 0\,.$$

   where $r^2 = (x - x_0)^2 + \nu\,(t - t_0)$ is the distance from a suitably chosen point $(x_0,\, t_0)$ on the axis of a space-time parabolic cylinder. The r^{-1} lower bound on $|\boldsymbol{u}|$ suggests a minimal rate of approach to the the CKN singular set **corresponding to $h = -1$.**

3. Our lower bound $h \geq -2/3$ keeps solutions away from the singular set.