OPEN PROBLEMS
AT BANFF
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Problem 1. Let $\mathfrak{S}_n$ be the permutation group on $[n]$. Given the pattern $\sigma = k(k-1)\cdots 321$, let $I_n(\sigma)$ be the number of *involutions* in $\mathfrak{S}_n$ that **avoid the pattern $\boldsymbol{\sigma}$**. Amitai Regev proved the case $k = 4$:

$$I_n((4321)) = \sum_{k\geq 0} \binom{n}{2k} C_k$$

which are the Motzkin numbers. Here C_k are the Catalan numbers.

Given an integer partition λ, draw the Young diagram and fill in hook-lengths of each cell. If a number t is not among these hook-lengths then λ is called a ***t*-core**. If it misses a, b, c then call it an **(a, b, c)-core partition**.

Let $N(n, n+1, n+2)$ be the number of all partitions that are $(n, n+1, n+2)$-core partitions. Then, Amdebrehan-Leven proved

$$N(n, n+1, n+2) = \sum_{k\geq 0} \binom{n}{2k} C_k.$$

Question. Is there a direct bijection between the above (4321)-avoiding involutions in $\mathfrak{S}_n$ and $(n, n+1, n+2)$-core partitions?

Problem 2. Let $[n]_q = \frac{1-q^n}{1-q}$ and $[n]!_q = [1]_q \cdots [n]_q$. The MacMahon's **$q$-Catalan polynomial** is

$$C_n(q) = \frac{1}{[n+1]_q} \binom{2n}{n}_q = \frac{[2n]!_q}{[n+1]!_q \, [n]!_q}.$$

William Chen (2015) conjectured that $C_n(q)$ **strictly convex functions**, i.e. $C_n''(q) > 0$ for $n \geq 2$.

I have an **almost proof** of this except the case $\boldsymbol{-1 < q < 0}$.

Question. For $0 < x < 1$ and $n \geq 2$, can you prove that

$$W_n(x) = \log \left(\frac{(1 + x^{4n-1})(1 + x^{2n})\boldsymbol{(1 - x^{2n+1})}}{(1 + x^{2n+1})(1 - x^{2n+2})} \right)$$

is a convex function of x? **Typo corrected in bold.**

Problem 3. the **hook-length** $h^\lambda(u) = \lambda_i + \lambda'_j - i - j + 1$ and **content** $c^\lambda(u) = j - i$ of a cell $u = (i, j)$ of a Young diagram of shape λ. The dimension of the irreducible representation of $GL(n, \mathbb{C})$ corresponding to λ with $\ell(\lambda) \leq n$ is given by

$$\dim_{GL}(\lambda, n) = \prod_{u \in \lambda} \frac{n + c^\lambda(u)}{h^\lambda(u)}.$$

Nekrasov-Okounkov's **hook-length formula**

$$\sum_{n \geq 0} q^n \sum_{\lambda \vdash n} \sum_{u \in \lambda} \frac{t + (h^\lambda(u))^2}{(h^\lambda(u))^2} = \prod_{k \geq 1} \frac{1}{(1 - q^k)^{t+1}}.$$

R. Stanley's **hook-content identity**

$$\sum_{n \geq 0} q^n \sum_{\lambda \vdash n} \sum_{u \in \lambda} \frac{t + (c^\lambda(u))^2}{(h^\lambda(u))^2} = \frac{1}{(1 - q)^t}.$$

For the irreducible representations of the **symplectic group** $\boldsymbol{Sp(2n)}$, the **symplectic content** of $u \in \lambda$ is

$$c^\lambda_{sp}(u) = \begin{cases} \lambda_i + \lambda_j - i - j + 2 & \text{if } i > j \\ i + j - \lambda'_i - \lambda'_j & \text{if } i \leq j. \end{cases}$$

Question. Can you prove this?

$$\sum_{n \geq 0} q^n \sum_{\lambda \vdash n} \sum_{u \in \lambda} \frac{t + (c^\lambda_{sp}(u))^2}{(h^\lambda(u))^2} = \prod_{k \geq 1} \frac{1}{(1 - q^{4k-2})(1 - q^k)^t}.$$

Remark. Cases $t = 0$ and $t = -1$ done (Amdeberhan-Andrews-Ballantine).

Problem 4. Let

$$F(t,x,z) := \prod_{j=0}^{\infty} \frac{1}{(1-tx^j)^{z-1}}.$$

(a) If $z=2$ then on the one hand we get Euler's

$$F(t,x,2) = \sum_{n\geq 0} \frac{(-1)^n x^{\binom{n}{2}}}{(x;x)_n} t^n,$$

on the other we get Pólya's formula (the "cycle index decomposition")

$$F(t,x,2) = \sum_{n\geq 0} Z(S_n, (1-x)^{-1}, \ldots, (1-x^n)^{-1}) t^n.$$

(b) If $t=x$ then we get Nekrasov-Okounkov's

$$F(x,x,z) = \sum_{n\geq 0} x^n \sum_{\lambda\vdash n} \prod_{\square\in\lambda} \left(1 - \frac{z}{h_\square^2}\right).$$

where $h_\square$ is the hook-length of a cell.

Question. Is there a unifying combinatorial right-hand side in

$$\prod_{j\geq 0} \frac{1}{(1-tx^j)^{z-1}} =?$$

Problem 5. Given a sequence of **positive numbers** $(a_k)_{k\geq 0}$, define the operator $\mathcal{L}a_k = a_k^2 - a_{k-1}a_{k+1}$. We say $(a_k)_k$ is **log-concave** provided that $\mathcal{L}a_k \geq 0$ for all $k \geq 0$.

If, after a repeated action of the operator $\mathcal{L}$, we find $\mathcal{L}^i a_k \geq 0$ for $1 \leq i \leq m$ and for all k, then $(a_k)_k$ is named ***m*-fold log-concave**. The sequence is called **infinitely log-concave** if $\mathcal{L}^i a_k \geq 0$ for all $i \geq 1$.

Given a graph G and x distinct colors, denote the number of proper colorings by $\kappa_G(x)$, referred as the **chromatic polynomial of *G***.

Theorem (June Huh).

The absolute values of coefficients in $\kappa_G(x)$, of any graph G, are log-concave.

Question. Are the (absolute values) coefficients of any chromatic polynomial infinitely log-concave?

Problem 6. The "quantum" version **qTSPP** of the number of *totally symmetric plane partitions*, contained in the cube $[0,n]^3$, is enumerated by

$$f_n(q) := \prod_{j=1}^{n} \prod_{k=1}^{j} \prod_{\ell=1}^{k} \frac{1-q^{j+k+\ell-1}}{1-q^{j+k+\ell-2}}.$$

L'Hôpital $f_n(1) = \lim_{q\to 1} f_n(q)$ restores the classical version $\prod_{1\leq \ell\leq k\leq j\leq n} \frac{j+k+\ell-1}{j+k+\ell-2}$.

Although $f_n(-1) = 0$ trivially, when n is odd, I observe the case n even is decidedly striking; namely that,

$$f_{2n}(-1) = \lim_{q\to -1} f_{2n}(q) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!},$$

the number A_n of $n \times n$ Âlternating Sign Matrices **ASMs**.

Question. Is there a non-analytic (more conceptual) reason for this connection between **qTSPP** and **ASMs**?

Problem 7. Consider the rational functions (in fact, polynomials)

$$F_n(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} (-q)^k \frac{2k+1}{n+k+1} \binom{2n}{n-k} \prod_{j=0,\, j\neq k}^{n} \frac{1+q^{2j+1}}{1+q}.$$

The numbers $\frac{2k+1}{n+k+1}\binom{2n}{n-k}$ belong to a family of **Catalan triangle** of which the special case $k=0$ yields the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Of further interest is $F_n(1) = E_{2n}$ the **Euler numbers**.

Question. Is it true that $F_n(q)$ has non-negative coefficients?

Problem 8. Given a Laurent polynomial g, let $CT(g)$ denote its **constant term**.

Consider the specific Laurent polynomial

$$f_n(x_1, \ldots, x_r) = \left(1 + \prod_{j=1}^{r}(1 + x_j) + \prod_{j=1}^{r}\left(1 + \frac{1}{x_j}\right)\right)^n .$$

Question. Is there a *Combinatorial Nullstellensatz type* (of Alon Noga) proof of this fact:

$$CT(f_n) = \sum_{m=0}^{n} \binom{n}{m} \sum_{k=0}^{m} \binom{m}{k}^{r+1} .$$

Problem 9 Let $\lambda_n = (n, n-1, \ldots, 2, 1)$ be the **staircase partition** and its **Young diagram** Y_n.

Question. In how many different ways a_n can one tile Y_n using monomers (1×1 squares) and dimers (1×2 or 2×1 rectangles)? Is there a determinant (Pfaffian) formulation of this enumeration, in **Kasteleyn's style**?

Problem 10. If $0 \leq k \leq< b$ are integers, prove the **coefficient-wise inequality**

$$\binom{a}{k}_q \binom{a+b}{b-k}_q \geq \binom{b}{k}_q \binom{a+b}{a-k}_q$$

or equivalently

$$\binom{a}{k}_q \binom{b}{k}_q \binom{a+b}{b}_q \left[\frac{1}{\binom{a+k}{k}_q} - \frac{1}{\binom{b+k}{k}_q} \right] \geq 0.$$