Descent-weighted trees and permutations

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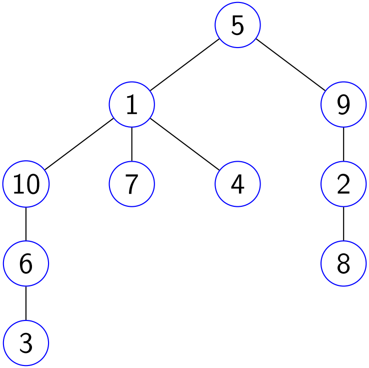
Rooted labelled trees

In a rooted labelled tree, all vertices have a unique label in $\{1, 2, \ldots, n\}$.



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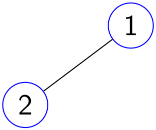

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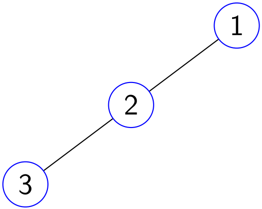

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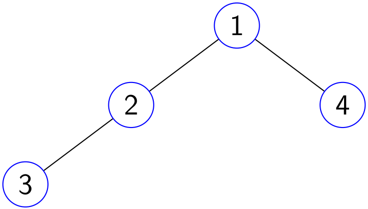

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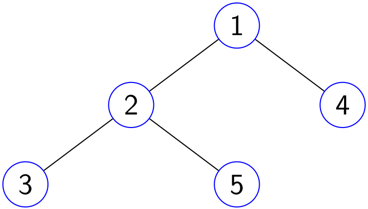

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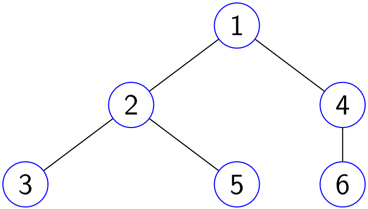

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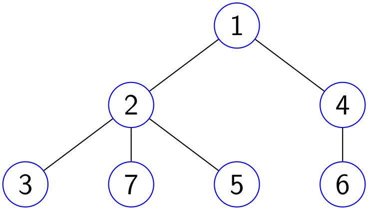

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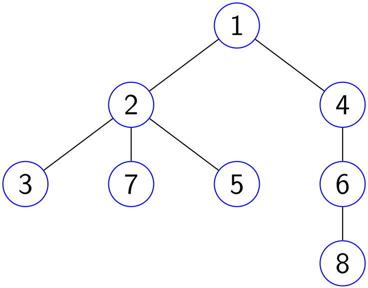

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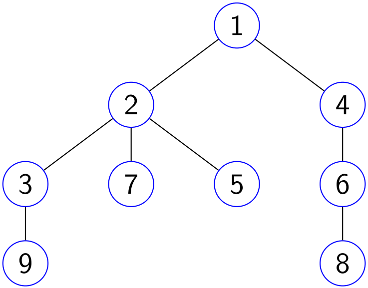

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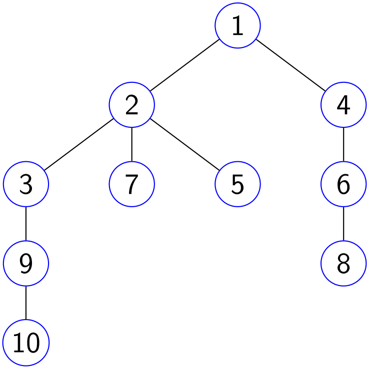

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Motivation

We would like a model of random trees that *interpolates* between uniformly random rooted labelled trees and recursive trees.

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This is achieved by defining a *weight* based on *descents*.

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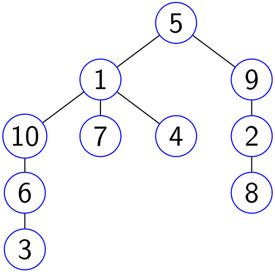

The example has four descents: $(5, 1)$, $(9, 2)$, $(6, 3)$ and $(10, 6)$.

The model

Let q be a positive real number. We consider random rooted labelled trees with n vertices whose probabilities are *proportional* to $q^{\text{number of descents}}$. The parameter q is allowed to depend on n.

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- Note that we obtain uniformly random rooted labelled trees for $q = 1$, random recursive trees as $q \to 0$ and random recursive trees with labels reversed as $q \to \infty$.
- Replacing q by $1/q$ amounts to reversing all labels. It is therefore enough to consider $q \leq 1$.

Descents in permutations

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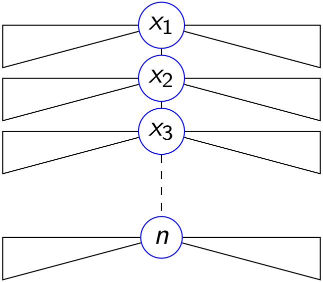
- This model is very similar to *Mallows permutations* (number of inversions) and *Ewens permutations* (number of cycles).

A connection between permutations and trees

Consider the path from the root to vertex n. The labels of the vertices on this path follow the descent-biased permutation model.



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3. If $q \to 0$ and $\log(1/q) = o(n)$, then

$$\frac{\log(1/q)}{n}(\pi_1, \ldots, \pi_r) \xrightarrow{d} (E_1, E_1 + E_2, \ldots, E_1 + \cdots + E_r),$$

where the E_i are i.i.d. $\mathrm{Exp}(1)$-variables.

Permutations: phases and local limits

4. If q is constant, then define the following Markov process: X_1 has density $\frac{\log(1/q)}{1-q} q^x$ on $[0,1]$, and for all $j \geq 1$, X_{j+1} has density

$$\frac{q^{x-X_j}}{\int_0^{X_j} q^z dz + \int_{X_j}^1 q^{z+1} dz} \left(q + (1-q)\mathbb{1}_{x \geq X_j}\right),$$

also on $[0,1]$. Then

$$\frac{1}{n}(\pi_1, \ldots, \pi_r) \xrightarrow{d} (X_1, X_2, \ldots, X_r).$$

Some proof ideas

In the *degenerate case* ($\log(1/q) \sim cn$), we can use direct counting: the number of permutations of $\{1, 2, \ldots, n\}$ with k descents is asymptotically equal to $(k+1)^n$, so the total weight of $\sim (k+1)^n q^k$ is maximal for k maximizing $\log(k+1) - ck$.

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In particular, if $q2^n \to 0$, then the weight of the identity permutation is greater than that of all others combined.

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If $P_n^k(q)$ is the weighted number of permutations of $\{1, 2 \ldots, n\}$ whose first element is k, and $S(x, y) = \sum_{n,k} P_n^k(q) \frac{x^n y^k}{n!}$, then

$$\frac{\partial}{\partial x} S(x, y) = (1-q) \frac{y}{1-y} \left(\frac{1}{e^{(q-1)x} - q} - \frac{qy}{e^{(q-1)xy} - q} - S(x, y) \right).$$

This can be used to analyze the moments of the first element.

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For constant q, the root degree has a discrete limit distribution (if $q \to 0$, it goes to infinity), with probabilities given by

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Moreover, we have a local limit, i.e., the distribution of the neighbourhood of radius r around the root converges for every fixed r.

Trees: root component

By *root component*, we mean the largest subtree containing the root that forms an increasing tree. Let $R_n(q)$ be the size of this component.

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- For fixed q, $R_n(q)$ converges weakly to a geometric random variable $\mathrm{Geom}(q^{1/(1-q)})$.
- If $q \to 0$, but $qn \to \infty$, scaling with q gives a limit: $qR_n(q) \xrightarrow{d} \mathrm{Exp}(1)$.

Trees: distances

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If q is fixed (and probably if $qn \to \infty$), the average distance of a random vertex from the root is asymptotically equal to $\frac{\log(1/q)}{1-q}\sqrt{\pi qn/2}$.

Some further directions

- "Mesoscopic" limit of permutations: if one considers descent-weighted permutations in windows of size $\Theta(\log(1/q_n))$, the number of descents is of constant order, and one observes a "diagonal pattern".

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- Changing the weight: instead of descents, one could also use *inversions* in trees.