BIRS OPEN PROBLEMS ABOUT FRIENDS-AND-STRANGERS GRAPHS

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Suppose $X = (V(X), E(X))$ and $Y = (V(Y), E(Y))$ are two simple graphs, each of which has n vertices. Imagine that the vertices of X are chairs and the vertices of Y are people. Two people in Y are adjacent if and only if they are friends with each other. There are $n!$ different ways to arrange the people in the chairs (with every chair occupied by exactly 1 person). Starting with such an arrangement of people in chairs, we can perform a *friendly swap* by swapping the positions of two people who are sitting in adjacent chairs *and* who are friends with each other. The *friends-and-strangers graph* of X and Y, denoted $\mathsf{FS}(X, Y)$, is the graph whose vertices are the arrangements of people in chairs, where two arrangements are adjacent whenever one is obtained from the other by a friendly swap. These graphs were originally introduced in [2]

More formally, we can think of the vertices of $\mathsf{FS}(X, Y)$ as the bijections from $V(X)$ to $V(Y)$. Two such bijections σ and τ are adjacent if and only if there exist $a, b \in V(X)$ such that $\sigma(a) = \tau(b)$, $\sigma(b) = \tau(a)$, and $\sigma(c) = \tau(c)$ for all $c \in V(X) \setminus \{a, b\}$. This description makes it clear that $\mathsf{FS}(X, Y)$ and $\mathsf{FS}(Y, X)$ are isomorphic; the isomorphism is given by $\sigma \mapsto \sigma^{-1}$.

As an example, suppose $V(X) = V(Y) = [n]$. Then the vertex set of $\mathsf{FS}(X, Y)$ is the symmetric group S_n. Each edge $\{i, j\}$ in $E(X)$ corresponds to the transposition $(i\ j)$ in S_n. If Y is a complete graph, then $\mathsf{FS}(X, Y)$ is the Cayley graph of S_n generated by transpositions corresponding to the edges in X.

Theorem 0.1 (Alon–Defant–Kravitz [1]; generalized by Wang–Chen [3])**.** *Fix some small $\varepsilon > 0$. Let X and Y be independently-chosen Erdős–Rényi random graphs in $\mathcal{G}(n, p)$, where $p = p(n)$ depends on n. If*

$$p \leq \frac{2^{-1/2} - \varepsilon}{n^{1/2}},$$

then $\mathsf{FS}(X, Y)$ has an isolated vertex (and is therefore disconnected) with high probability. If

$$p \geq \frac{\exp(2(\log n)^{2/3})}{n^{1/2}},$$

then $\mathsf{FS}(X, Y)$ is connected with high probability.

Problem 0.2. *Understand $\mathsf{FS}(X, Y)$ when X and Y are independent Erdős–Rényi random graphs in $\mathcal{G}(n, p)$. If*

$$p \leq \frac{2^{-1/2} - \varepsilon}{n^{1/2}},$$

then how many connected components does $\mathsf{FS}(X, Y)$ have? If

$$p \geq \frac{\exp(2(\log n)^{2/3})}{n^{1/2}},$$

then what is the diameter of $\mathsf{FS}(X, Y)$? What are the minimum and maximum degrees of $\mathsf{FS}(X, Y)$? Can we say "how connected" $\mathsf{FS}(X, Y)$ is?

Problem 0.3. *Fix X and Y, and consider the Markov chain whose state space is the vertex set of* $\mathsf{FS}(X,Y)$ *where at each step, we choose two people (or maybe it's better to choose two friends) at random and swap them if they are allowed to swap (i.e., they are friends and are sitting in adjacent chairs). Here, maybe we would want to fix Y to be some specific graph like a path or a cycle (or the complement of a path or a cycle). Or maybe it's interesting to consider when X and Y are independent Erdős–Rényi random graphs in $\mathcal{G}(n,p)$.*

References

[1] N. Alon, C. Defant, and N. Kravitz, Typical and extremal aspects of friends-and-strangers graphs. To appear in *J. Combin. Theory Ser. B*, (2022).

[2] C. Defant and N. Kravitz, Friends and strangers walking on graphs. *Comb. Theory*, **1** (2021).

[3] L. Wang and Y. Chen, Connectivity of friends-and-strangers graphs on random pairs. arXiv:2208.0080.