Enumerative and Analytic Combinatorics from Pop-Stack Sorting

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Analytic and Probabilistic Combinatorics BIRS Workshop

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Theorem (Ungar, 1982)

The maximum number of iterations of Pop *needed to send a permutation in* S_n *to the identity is* $n-1$.

Conjecture (D., 2020)

The average number of iterations of Pop *needed to sort a random permutation in* S_n *is* $n(1-o(1))$.

(I can prove that the average is at least $n/2$. Can you prove that it is at least $0.5001n$?)

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Theorem (Claesson–Guðmundsson, 2019)

For each fixed $t \geq 0$, the generating function for t-pop sortable permutations is rational.

Pop-Stacked Permutations

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The structure of $\mathsf{Pop}(S_n)$ was studied by Asinowski–Banderier–Hackl and by Asinowski–Banderier–Billey–Hackl–Linusson.

Claesson–Guðmundsson–Pantone gave a polynomial-time algorithm for computing $|\mathsf{Pop}(S_n)|$ and used it to compute these numbers for $n \leq 1000$.

Meet-Semilattices

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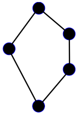

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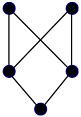

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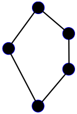

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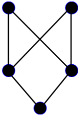

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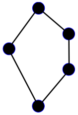

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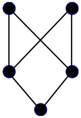

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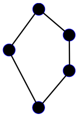

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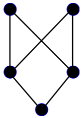

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Write $\bigwedge X$ for the meet of a set $X \subseteq L$.

A *lattice* is a meet-semilattice whose dual is also a meet-semilattice.

The Weak Order on S_n

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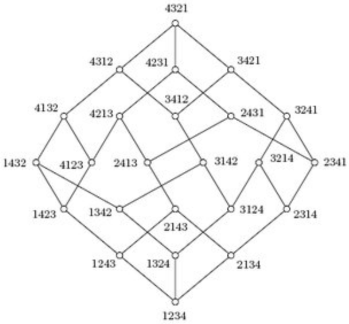
For $\pi, \sigma \in S_n$, write $\pi \lessdot \sigma$ if π can be obtained from σ by reversing a single descent. For example, $346251 \lessdot 364251$. These relations form the cover relations of the *weak order* on S_n.



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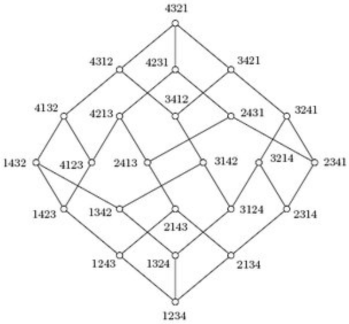

The weak order is a lattice.

Pop-Stack Sorting for Meet-Semilattices

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Definition (D., 2022)

Given a meet-semilattice L, define the *pop-stack sorting operator* $\mathsf{Pop}\colon L \to L$ by

$$\mathsf{Pop}(x) = \bigwedge\{y \in L : y \lessdot x\}.$$

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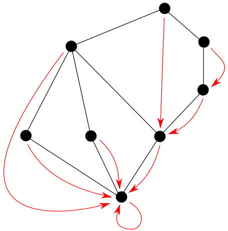

Ungar's Theorem for Coxeter Groups

Let W be a finite Coxeter group with Coxeter number h. The weak order on W is a lattice.

Theorem (D., 2022)

The maximum number of iterations of Pop *needed to send an element of W to the identity is $h - 1$.*

Rational Generating Functions in Other Types

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Theorem (D., 2022)

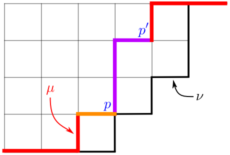
For each fixed $t \geq 0$, the generating function that counts t-pop sortable elements of the hyperoctahedral group B_n is rational.

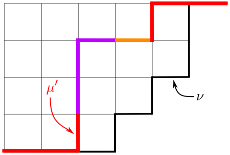
Theorem (D., 2022)

For each fixed $t \geq 0$, the generating function that counts t-pop sortable elements of the affine symmetric group $\widetilde{A}_n$ is rational.

ν-Tamari Lattices

Fix a lattice path ν. Let $\mathrm{Tam}(\nu)$ be the set of lattice paths lying weakly above ν. Make $\mathrm{Tam}(\nu)$ into a lattice with the following cover relations $\mu \lessdot \mu'$:





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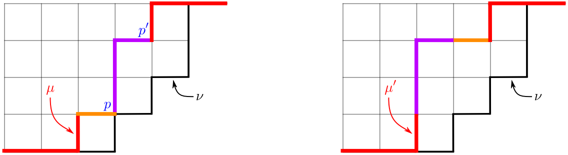

$\mathrm{Tam}((\mathrm{NE}^m)^n)$ is the n-th *m-Tamari* lattice $\mathrm{Tam}_n(m)$.

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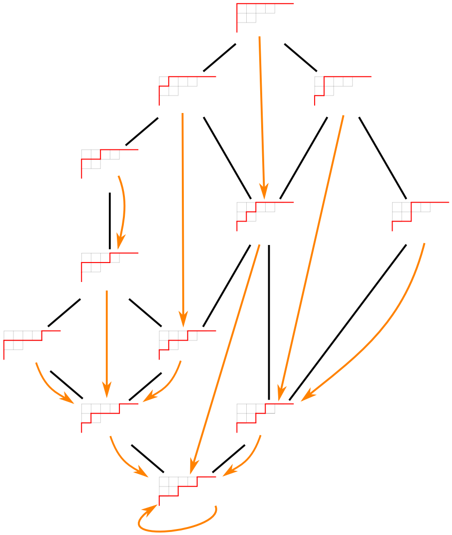
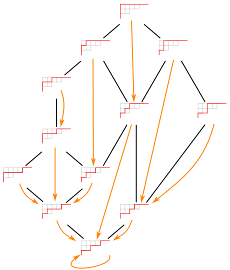


Pop-Stack Sorting on m-Tamari Lattices



Let $M(\mathrm{Tam}(\nu))$ be the maximum number of iterations of Pop needed to send every element of $\mathrm{Tam}(\nu)$ to $\widehat{0} = \nu$.

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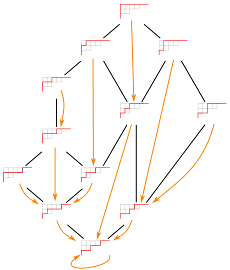

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I have computed $M(\mathrm{Tam}(\nu))$ for all ν.

Theorem (D., 2022)

We have $M(\mathrm{Tam}_n(m)) = m + n - 2$ *and*

$$N(\mathrm{Tam}_n(m)) = \frac{1}{n-1}\binom{(m+1)(n-2)+m-1}{n-2}.$$

In particular, $N(\mathrm{Tam}_n) = C_{n-2}$.

More Rational Generating Functions

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Let $h_t(m,n)$ be the number of t-pop sortable elements of $\mathrm{Tam}_n(m)$.

Conjecture (D., 2022)

For fixed $t, m \geq 1$, the generating function $\sum_{n\geq 1} h_t(m,n)x^n$ is rational.

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Theorem (Hong, 2022)

The above conjecture is true when $m = 1$. In fact,

$$\sum_{n\geq 1} h_t(1,n)x^n = \frac{x}{1-2x-\sum_{j=2}^{t} C_{j-1}x^j}.$$

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Nathan Williams and I introduced a much broader family of lattices called *semidistrim lattices*. We described how to define a bijective rowmotion operator on any semidistrim lattice.

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Let L be semidistrim, and let L^* be its dual. Then

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The equality $|\mathsf{Pop}(L)| = |\mathsf{Pop}(L^*)|$ does **not** hold for arbitrary finite lattices.

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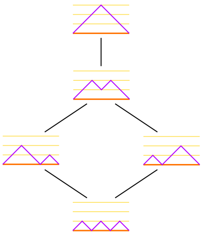

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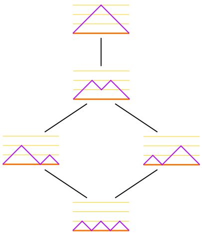

Theorem (Sapounakis–Tasoulas–Tsikouras, 2006)

Then

$$|\mathsf{Pop}(\mathcal{L}_n)| = \sum_{k=0}^{n+1} \frac{1}{k+1}\binom{2k}{k}\binom{n+k+1}{3k}.$$

More Pop Images

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Nathan Williams and I stated several enumerative conjectures about the image of Pop for Tamari lattices, the weak order of B_n, lattice of order ideals of root posets, type-B Tamari lattices, and bipartite Cambrian lattices. All but the last were resolved in a recent preprint by Yunseo Choi and Nathan Sun.

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Question

What is $|\mathsf{Pop}(\mathrm{Tam}_n(m))|$ when $m \geq 2$?

Infinite Meet-Semilattices

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Problem

Let L be an interesting infinite meet-semilattice. Let a_t be the number of t-pop sortable elements of L. What is $\sum_{t\geq 0} a_t x^t$? Is it rational?

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Potential candidates for L include

- The weak order of an infinite Coxeter group such as the affine symmetric group.
- The affine Tamari meet-semilattice (312-avoiding affine permutations under the weak order).
- The weak order on the positive braid monoid.

Thank You!