Cutting trees revisited

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Workshop "Analytic and Probabilistic Combinatorics",
BIRS, Banff, Canada, 17.11.2022

Cutting down procedure

- Meir & Moon [1970, 1974]:

Cutting down procedure for rooted trees

1. Take a rooted tree T
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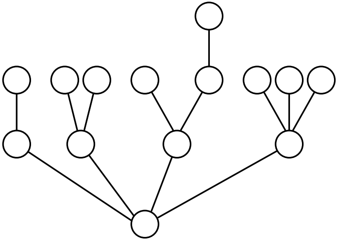

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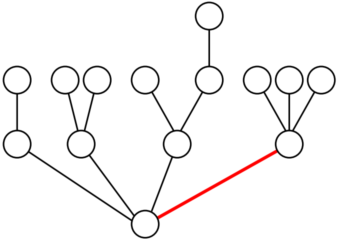

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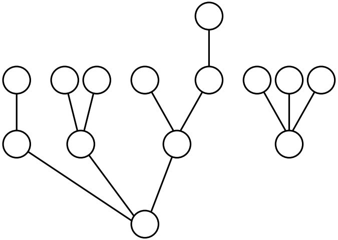

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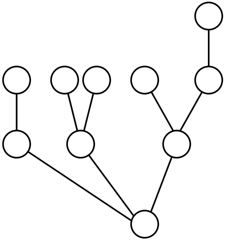

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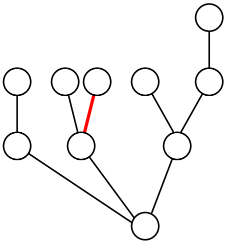

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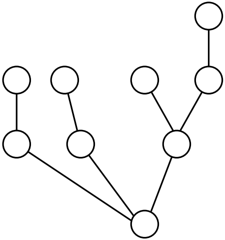

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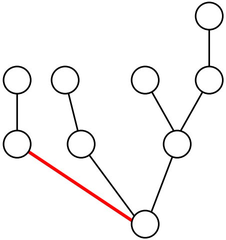

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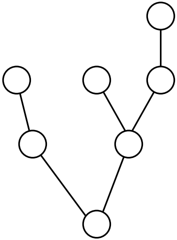

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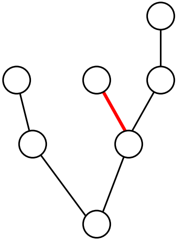

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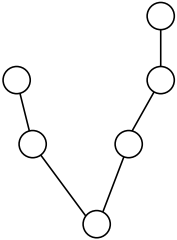

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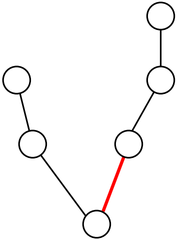

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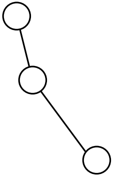

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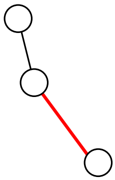

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Number of cuts to isolate root

Meir & Moon (1970, 1974): X_n, number of cuts of size-n tree for two random tree models:

- random Cayley-trees (= rooted labelled trees)
- random recursive trees (= increasingly labelled trees)

Start the cutting down procedure with random size-n tree
→ tree models behave quite different

Expectation/variance for Cayley-trees:

$$\mathbb{E}(X_n) \sim \sqrt{\frac{\pi n}{2}}, \qquad \mathbb{V}(X_n) \sim \left(1 - \frac{\pi}{2}\right) \cdot n$$

Expectation/variance for recursive trees:

$$\mathbb{E}(X_n) \sim \frac{n}{\log n}, \qquad \mathbb{E}(X_n^2) \sim \frac{n^2}{\log^2 n} \quad \Rightarrow \quad \mathbb{V}(X_n) = o\left(\frac{n^2}{\log^2 n}\right)$$

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Recursive approach

Used recursive description:

$$X_n \stackrel{(d)}{=} 1 + X_{S_n}, \qquad S_n : \text{size of subtree containing root}$$

Size of remaining subtree:

Cayley-trees: $\mathbb{P}\{S_n = m\} = \binom{n}{m} \frac{m^m (n-m)^{n-m-1}}{(n-1) n^{n-1}}$

Recursive trees: $\mathbb{P}\{S_n = m\} = \frac{n}{(n-1)(n-m)(n-m+1)}$

G. f. **treatment of recurrences** for first moments yields results

Limitation of approach: only applicable (in direct way) if randomness is preserved for remaining tree

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- Pan (2003, 2004, 2006):
  - **characterization of simply gen. tree families** (= cond. GW-trees) satisfying randomness preservation property
    - Cayley-trees
    - d-ary trees
    - generalized ordered trees
  - **Rayleigh limiting distribution of X_n**
    for such "very simple tree families"

$$\frac{X_n}{\sqrt{n}} \xrightarrow{(d)} \text{Rayleigh}(\sigma), \quad \text{density} \quad f_\sigma(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0$$

  - Cutting down non-crossing trees → Rayleigh-limit law
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each cut costs a toll depending on size of the tree

→ study total costs of one-sided and two-sided destruction of "very simple trees"

→ limiting distribution results via method of moments

- Pan & Kuba (2007): application of two-sided destruction to **analysis of Union-Find-algorithms** (maintaining set partitions)

- Drmota, Iksanov, Möhle & Rösler (2009):

stable limit law of number of cuts X_n **for recursive trees**

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characteristic fct. $\varphi_Y(t) = e^{it \log|t| - \frac{\pi}{2}|t|}$

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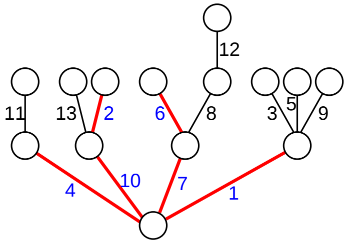
  - **Description of cutting procedure via records** in edge-labelled trees
    Record: edge-label smaller than labels of all ancestor-edges



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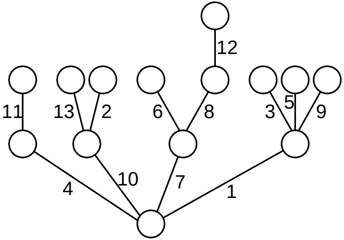

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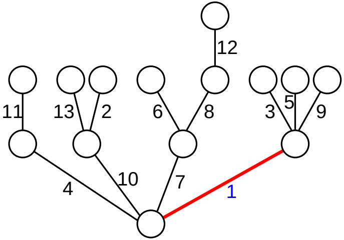

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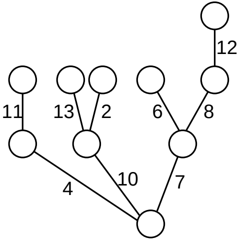

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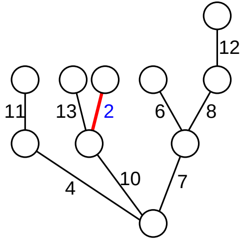

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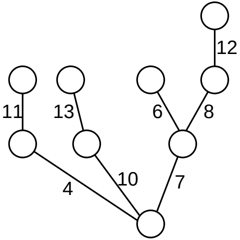

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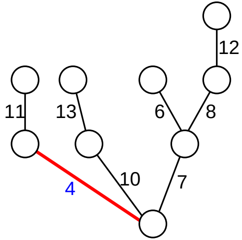

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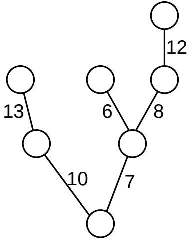

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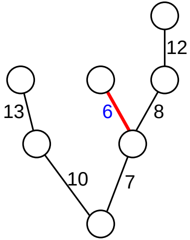

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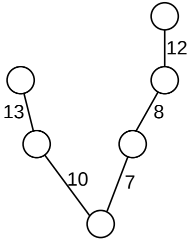

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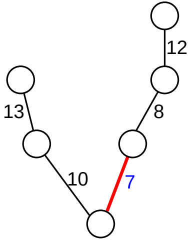

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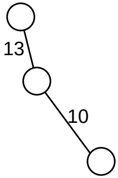

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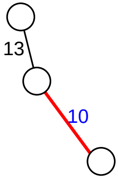

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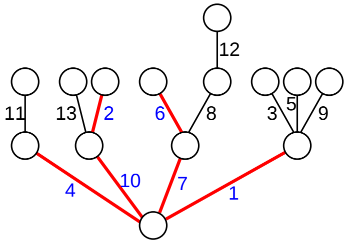

Cut ↔ Edge-record

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k-cut model for rooted trees

- Cai, Devroye, Holmgren & Skerman (2019)
  Berzunza, Cai & Holmgren (2020, 2021):

Adapting cutting down procedure:

A vertex has to be cut k-times
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Considered tree models:

- paths and "path-like trees"
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k-cuts in paths

$k = 1$: Cutting down path:

$X_n \overset{(d)}{=}$ number of records in sequence of n i.i.d. Unif$[0,1]$ r.v.

$\overset{(d)}{=}$ number of left-to-right maxima/minima in random permutation

$\overset{(d)}{=}$ number of cycles in random permutation

Limiting behaviour of X_n:

Goncharov (1942); Shepp-Lloyd (1966):

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First two moments:

$$\mathbb{E}(X_n) \sim \eta_k n^{1-\frac{1}{k}}, \qquad \mathbb{E}(X_n^2) \sim \gamma_k n^{2-\frac{2}{k}},$$

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Recursive approach

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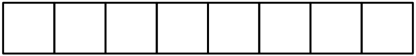

For recursive approach **have to take care of auxiliary quantity:**
number of nodes already cut once

→ **"Urn model"** with non-deterministic ball replacement scheme:

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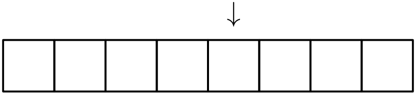

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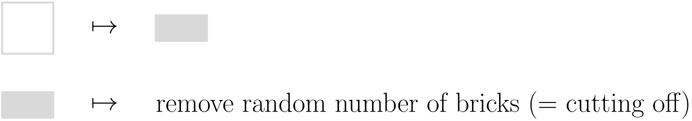

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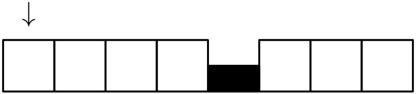

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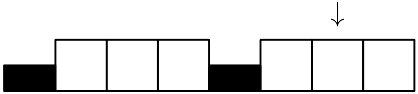

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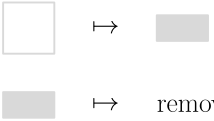

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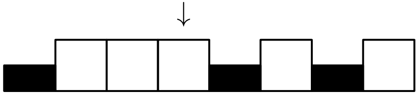

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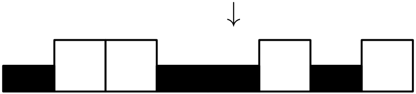

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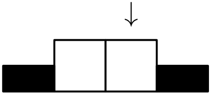

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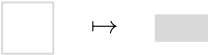

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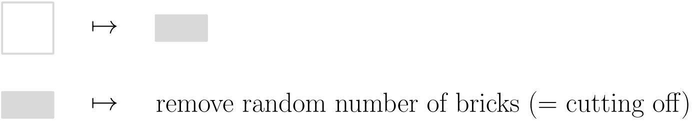

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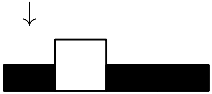

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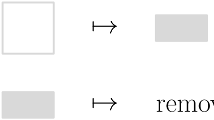

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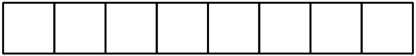

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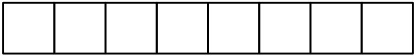

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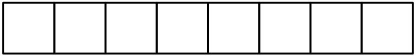

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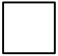 $\mapsto$ 

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Stochastic recurrence

$\tilde{X}_{n,j}$: number of cuts to destroy path of length n
starting with j random nodes already cut once

Distributional recurrence:

$$\tilde{X}_{n,j} \stackrel{(d)}{=} V_{n,j} \cdot \tilde{X}_{n,j+1} + (1 - V_{n,j}) \cdot \tilde{X}_{S_1,S_2}, \quad 0 \le j \le n, n \ge 1, \quad \tilde{X}_{0,0} = 0,$$

where $V_{n,j} \stackrel{(d)}{=} \text{Bernoulli}(1 - \frac{j}{n}),$

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Generating functions approach

probability generating function $\mathbb{E}(v^{\tilde{X}_{n,j}})$ → recurrence

suitable g.f. $\tilde{F} := \tilde{F}(z,x,v) := \sum_{n,j\geq 0} \binom{n}{j} \cdot \mathbb{E}(v^{\tilde{X}_{n,j}}) z^n x^j$

→ Linear first-order PDE:

$$z\tilde{F}_z = v\left(\tilde{F}_x + \frac{zx}{1-z(1+x)}\tilde{F}\right)$$

Explicit solution:

$$\tilde{F}(z,x,v) = e^{\int_x^\infty \frac{zte^{\frac{x-t}{v}}}{1-z(1+t)e^{\frac{x-t}{v}}}dt}$$

Solution of original problem: vanish auxiliary quantity $x = 0$:

$$F(z,v) := \tilde{F}(z,0,v) = \sum_{n\geq 1} \mathbb{E}(v^{X_n}) z^n$$

$$\Longrightarrow \quad F(z,v) = e^{\int_0^\infty \frac{zte^{-\frac{t}{v}}}{1-z(1+t)e^{-\frac{t}{v}}}dt}$$

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suitable g.f. $\tilde{F} := \tilde{F}(z,x,v) := \sum_{n,j\geq 0} \binom{n}{j} \cdot \mathbb{E}(v^{\tilde{X}_{n,j}}) z^n x^j$

→ Linear first-order PDE:

$$z\tilde{F}_z = v\left(\tilde{F}_x + \frac{zx}{1-z(1+x)}\tilde{F}\right)$$

Explicit solution:

$$\tilde{F}(z,x,v) = e^{\int_x^\infty \frac{zte^{\frac{x-t}{v}}}{1-z(1+t)e^{\frac{x-t}{v}}}dt}$$

Solution of original problem: vanish auxiliary quantity $x = 0$:

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Moments

Expectation: $$\mathbb{E}(X_n) = H_n + \sum_{\ell=1}^{n} \frac{Q(\ell)}{\ell},$$

with $$Q(n) = \sum_{\ell=0}^{n-1} \frac{(n-1)^{\underline{\ell}}}{n^{\ell}} = \int_0^\infty (1+\frac{x}{n})^{n-1} e^{-x} dx,$$ Ramanujan's Q-function

Asymptotics of m-th integer moments:

$$\mathbb{E}\left(\left(\frac{X_n}{\sqrt{n}}\right)^m\right) \sim \frac{m!}{\Gamma(1+\frac{m}{2})} \cdot [w^m] e^{\frac{\sqrt{2} w \arccos\left(-\frac{w}{\sqrt{2}}\right)}{\sqrt{1-\frac{w^2}{2}}}}, \quad m \geq 0$$

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Fréchet and Shohat moment conv. thm. $\Rightarrow \frac{X_n}{\sqrt{n}} \xrightarrow{(d)} X$,

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$$f(z) = \sum_{n\geq 0} f_n z^n \xrightarrow{\mathcal{B}_\alpha} \hat{f}(z) = \sum_{n\geq 0} f_n \frac{z^n}{\Gamma(1+\alpha n)}$$

$\mathcal{B}_\alpha(f(z)) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \frac{E_\alpha(zt)}{t} f(\frac{1}{t})dt,$ Mittag-Leffler-fct. $E_\alpha(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(1+\alpha n)}$

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$$M(s) = \frac{1}{2\pi i}\int_{C-\infty}^{C+i\infty} \left(1 + \frac{2}{\sqrt{\pi}}\int_0^{s^2t^2} e^{-\tau^2} d\tau\right) e^{s^2t^2 + \frac{\frac{\sqrt{2}}{t}\arccos(-\frac{1}{\sqrt{2}t})}{\sqrt{1-\frac{1}{2t^2}}}} dt, \quad \Re C > \frac{1}{\sqrt{2}}$$

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Recursive approach for general k

General k: adaptions for recursive approach

- Require $k-1$ auxiliary quantities:
  j_1 nodes cut once, ..., j_{k-1} nodes cut $(k-1)$-times
- "Urn model"-description with k types of balls
- Generating functions approach → linear first-order PDE:

$$z\tilde{F}_z = v\left(\tilde{F}_{x_1} + x_1\tilde{F}_{x_2} + x_2\tilde{F}_{x_3} + \cdots + x_{k-2}\tilde{F}_{x_{k-1}} + \frac{x_{k-1}z\tilde{F}}{1 - z(1 + x_1 + x_2 + \cdots + x_{k-1})}\right)$$

- PDE is explicitly solvable
- Vanishing all auxiliary variables $x_1, \ldots, x_{k-1} = 0$
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Limiting behaviour

Asymptotic behaviour of m-th integer moment:

$$\mathbb{E}\left(\left(\frac{X_n}{n^{1-\frac{1}{k}}}\right)^m\right) \sim \frac{m!}{\Gamma\left(1+\frac{(k-1)m}{k}\right)} \cdot [w^m]e^{\varphi(w)},$$

with **exponent:**

$$\begin{aligned}\varphi(w) &= \sum_{m=1}^{\infty} \frac{(k!)^{\frac{m}{k}}\Gamma(\frac{m}{k}+1)\Gamma(\frac{(k-1)m}{k})}{m!} w^m \\ &= k!w \int_0^{\infty} \frac{dx}{x^k - k!wx + k!} \\ &= \sum_{j=1}^{k} \frac{x_j w}{k-(k-1)x_j w} \ln(-x_j), \qquad x_j \text{ roots of } p(x) = x^k - k!wx + k!\end{aligned}$$

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Convergence in distribution $\frac{X_n}{n^{1-\frac{1}{k}}} \xrightarrow{(d)} X$,

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k-cutting trees

Berzunza, Cai & Holmgren (2020, 2021); Wang (2021):

Limiting distribution result for conditioned GW-trees:

$$\frac{X_n}{\sigma^{\frac{1}{k}} n^{1-\frac{1}{2k}}} \xrightarrow{(d)} X,$$

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Recursive approach:

- only applicable for "very simple trees"
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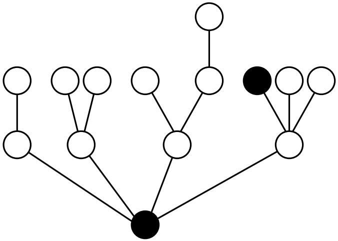
Isolating a set of nodes in trees

Cutting algorithm for isolating multiple nodes:

- Take a tree T with a distinguished set $S \subseteq V(T)$ of nodes
- Select a vertex/edge at random
- Remove vertex/edge and discard all subtrees not containing any vertex of S
- Iterate procedure and terminate when all nodes of S are isolated/removed



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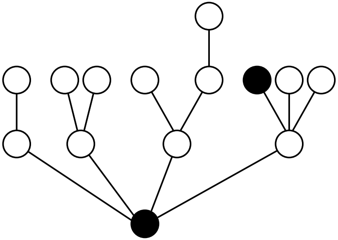

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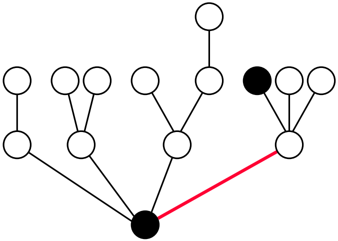

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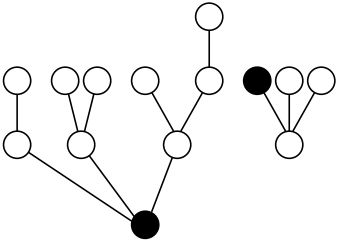

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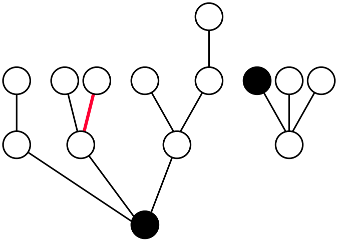

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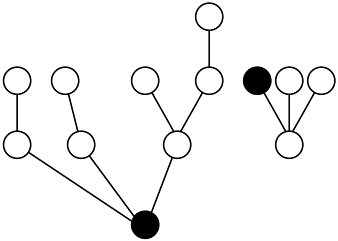

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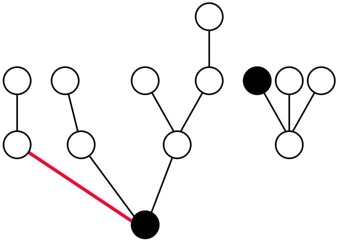

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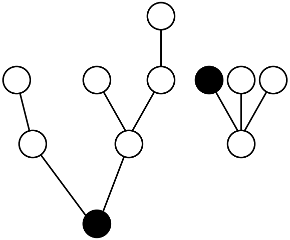

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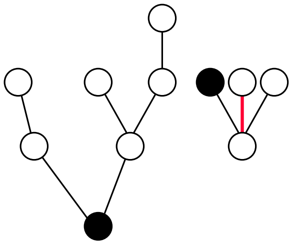

Isolating a set of nodes in trees

Cutting algorithm for isolating multiple nodes:

- Take a tree T with a distinguished set $S \subseteq V(T)$ of nodes
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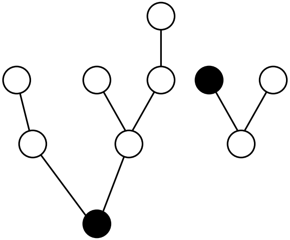

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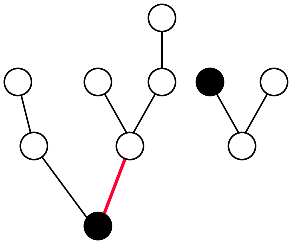

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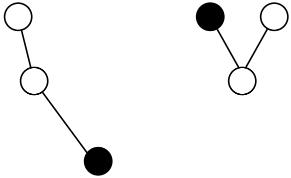

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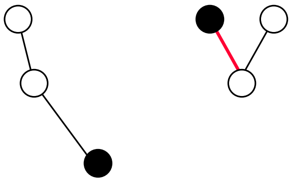

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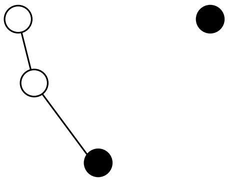

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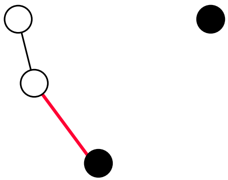

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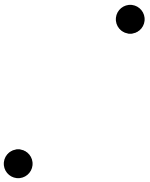

Previous studies for number of cuts

- Addario-Berry, Broutin & Holmgren (2014):
  isolating ℓ random nodes in Cayley-trees:

$$\frac{X_n^{[\ell]}}{\sqrt{n}} \xrightarrow{(d)} \chi_\ell,$$

χ_ℓ: chi-distributed r.v. with 2ℓ degrees of freedom,

density $f_\ell(x) = \frac{x^{2\ell-1}}{2^{\ell-1}(\ell-1)!} e^{-\frac{x^2}{2}}$, $x > 0$

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Consider multiple isolation in Cayley-trees:

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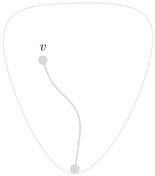
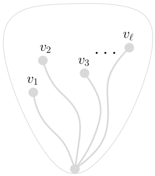


- **"random ancestor-tree":** all nodes on each path from root to ℓ random nodes



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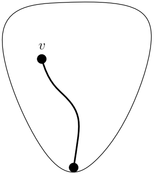

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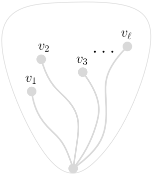

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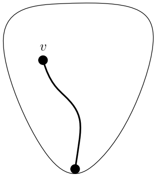

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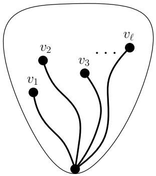

Recursive approach

$X_{n,\ell}$: **number of cuts** to isolate all nodes in ancestor-tree of ℓ random nodes

- Suitable g.f. $F(z,u,v) := \sum_{n,\ell} \frac{n^{n-1}}{n!} \binom{n}{\ell} \cdot \mathbb{E}(v^{X_{n,\ell}}) z^n u^\ell$
- $\rightarrow$ quasi-linear first-order PDE
- **Explicit solution:** ($T(x) = xe^{T(x)}$ tree function)

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Suitable to gain further results on multiple isolation
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- behaviour if number ℓ of nodes grows with size n

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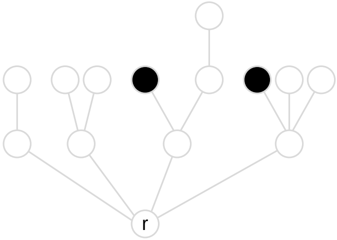
Specific case: separating a set $P \subseteq V(T)$ of nodes from root r in tree T

→ Stop cutting procedure to isolate root
if remaining subtree does not contain any node from P



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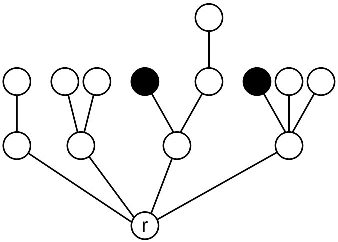

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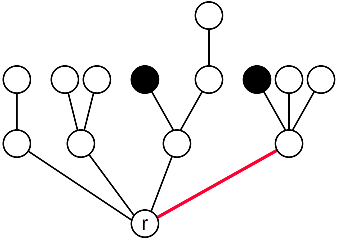

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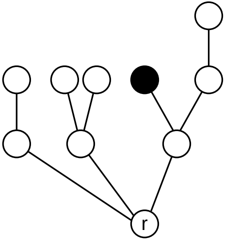

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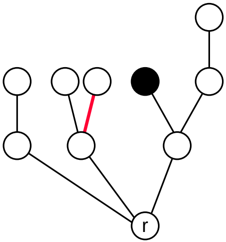

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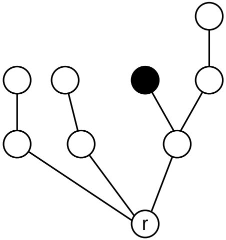

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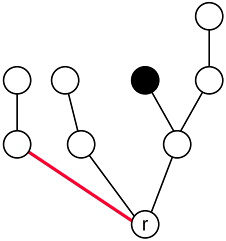

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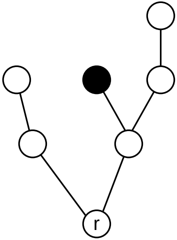

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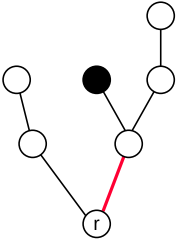

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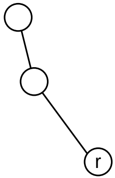

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Interesting quantities:

Y_n: number of cuts until all nodes from P are separated

R_n: size of the remainder tree when all nodes are separated

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Separating ℓ random nodes

Recursive approach → easily gives explicit solution for g.f.

$\ell = 1$: **separating a single node:** $\frac{Y_{n,1}}{\sqrt{n}} \xrightarrow{(d)} Y_1$,

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$$\text{density:} \quad f_1(x) = \int_x^\infty e^{-\frac{t^2}{2}}\,dt, \quad x > 0$$

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Separating all leaves

Recursive approach more involved:

→ requires auxiliary parameters

- $\sharp$ leaves that are "active" during cutting procedure (leaf has not been separated)
- $\sharp$ leaves that are "inactive" during cutting procedure (internal node in original tree)

→ **linear first-order PDE** involving evaluations of tree function

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Results for separating leaves

Size R_n of remainder tree: $R_n \xrightarrow{(d)} R$, R discrete law

Probability g.f. $p(v) = \mathbb{E}(v^R)$:

$$p(v) = 1 - \frac{1}{e}\int_0^1 \frac{1}{1-K(t)}dt$$
$$+ \frac{1}{e}\int_0^1 \frac{v(1-K) + (-1-v+e^{-1}tv(1-v)+e^{-1}K+vK^2)M + (2+v-vK)M^2 - M^3}{(1-K)(1-M)^3}dt,$$

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Probabilities for small remainder tree size/expectation:

$$\mathbb{P}\{R=0\} = 1 - \frac{1}{e}\int_0^1 \frac{1}{1-K(t)}dt \approx 0.462117, \quad \text{(separating = isolating)}$$

$$\mathbb{P}\{R=1\} = \frac{1}{e} - \frac{1}{e}\int_0^1 \frac{te^{-t}}{1-K(t)}dt \approx 0.217584,$$

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Results for separating leaves

Size R_n of remainder tree: $R_n \xrightarrow{(d)} R$, R discrete law

Probability g.f. $p(v) = \mathbb{E}(v^R)$:

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End of talk