Alberta-Montana Combinatorics and Algorithms Days at BIRS

A class of optimal constant weight ternary codes

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An overview of codes

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- The Hamming distance between the codewords is 4 ($d = 4$).
- The code has constant weight 4 ($w = 4$).
- We denote this as a $(5, 4, 4)_3$-code.
- How many codewords can we have with $n = 5$, $d = 4$, $w = 4$ (Maximize M)?

Upper bounds for M

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$$A_q(n,d,w) \leq \left\lfloor \frac{n(q-1)}{w} A_q(n-1,d,w-1) \right\rfloor \tag{1}$$

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- For $q = 3$, $n = 5$, $d = 4$ and $w = 4$ we have $a = 3(4)^2 - 2(2)(5)(4) + 2(5)(4) = 8 > 0$. Since the condition is met, we can use bound (2) to compute $A_3(5, 4, 4) \leq \lfloor \frac{2nd}{3w^2 - 4nw + 2nd} \rfloor = \frac{40}{8} = 5$.

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$$A_3(6,4,5) = \left\lfloor \frac{nd(q-1)}{qw^2 - 2(q-1)nw + nd(q-1)} \right\rfloor = \frac{48}{3} = 16$$

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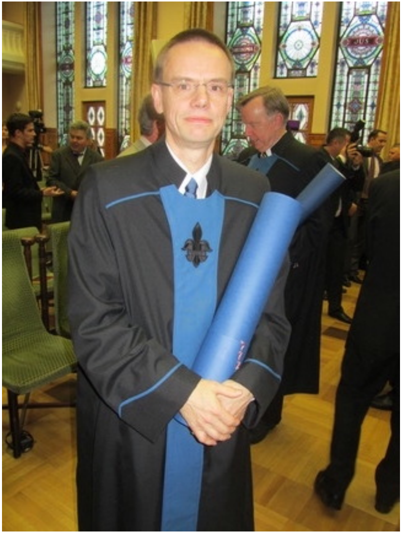
There are only 6 codewords and it is hard to find 10 more.

A 2002 Electronic Journal of Combinatorics result of



Patric *Östergård*

$\ddot{O}sterg\dot{a}rd's$ result

Showed that an optimal constant weight ternary code having length $n = 6$, distance $d = 4$ and constant weight $w = 5$ consists of 12 codewords:

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Theorem (E.J.C. 2002)

If $p \geq 3$ is a prime power and $m \geq 1$, then

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Let $p = 5$, $m = 1$, then $A_3(6, 4, 5) = 12$.

An extension of *Östergård*'s result

We begin by looking at the Derived part of W:

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We now use Johnson bound (1):

$$A_3(6,4,5) \leq \left\lfloor \frac{n(q-1)}{w} A_q(n-1,d,w-1) \right\rfloor = \frac{2(6)}{5}(5) = 12$$

and conclude that $A_3(6,4,5) \leq 12$.

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\end{array}$$

There we have the desired 12 codewords.

The extension

Theorem

Let C be a conference matrix of order $n+1$ (ie $W(n+1,n)$ with 0 diagonal). Then the rows of C and $-C$ together form an optimal constant weight ternary code and so

$$A_3(n+1, \frac{n+3}{2}, n) = 2(n+1).$$

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Corollary

If $p \geq 3$ is a prime power and $m \geq 1$, then

$$A_3\left(p^m+1, \frac{p^m+3}{2}, p^m\right) = 2(p^m+1).$$

Remark: There is a $W(16,15)$ and 15 is a composite number.

The new class of optimal ternary codes

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Next we try to show that for every odd prime power p and positive integer m:

Theorem

$$A_3\left(\frac{p^{m+1}-1}{p-1}, p^{m-1}\left(\frac{p+3}{2}\right), p^m\right) = 2\left(\frac{p^{m+1}-1}{p-1}\right).$$

We begin with the definition and some examples of *Orthogonal Arrays*

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Example

Let $n = 3$ and $m = 4$, then

$$O = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 3 & 3 \\ 2 & 1 & 2 & 3 \\ 2 & 3 & 1 & 2 \\ 2 & 2 & 3 & 1 \\ 3 & 1 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 3 & 3 & 2 & 1 \end{bmatrix}$$

is an $OA(3, 4)$ on $S = \{1, 2, 3\}$.

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O is an $OA(5,6)$ on $S = \{1,2,3,4,5\}$.

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O is an $OA(5,6)$ on $S = \{1,2,3,4,5\}$. The superimposition of any two distinct rows of O will have exactly one common symbol.

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$$\begin{matrix} 2 & 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 2 & 3 & 4 \end{matrix}$$

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5	5	4	3	2	1

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  contain each group element exactly $\lambda/|G|$ times.
- Namely, the conjugate inner product of any two distinct rows of W contains each element of G exactly $\lambda/|G|$ times.

Example of a BGW

Example of a BGW

Example

Let $v = 6$, $k = 5$ and $\lambda = 4$, then

$$W = \begin{bmatrix} 0 & 4 & 4 & 4 & 4 & 4 \\ 2 & 0 & 3 & 4 & 1 & 2 \\ 2 & 1 & 0 & 3 & 2 & 4 \\ 2 & 2 & 1 & 0 & 4 & 3 \\ 2 & 3 & 4 & 2 & 0 & 1 \\ 2 & 4 & 2 & 1 & 3 & 0 \end{bmatrix}$$

is a $BGW(6,5,4)$ over $\mathbb{Z}_4$.

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Example of a BGW

0	4	4	4	4	4
2	2	1	0	4	3

Example of a BGW

$$\begin{matrix} 0 & \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 \\ \alpha^2 & \alpha^2 & \alpha^1 & 0 & \alpha^4 & \alpha^3 \end{matrix}$$

Example of a BGW

$$\begin{matrix} 0 & \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 \\ \alpha^2 & \alpha^2 & \alpha^1 & 0 & \alpha^4 & \alpha^3 \end{matrix}$$

Example of a BGW

$$\begin{matrix} \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 \\ \alpha^2 & \alpha^1 & \alpha^4 & \alpha^3 \end{matrix}$$

Example of a BGW

$$\begin{array}{cccc} \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 \\ \alpha^{-2} & \alpha^{-1} & \alpha^{-4} & \alpha^{-3} \end{array}$$

Example of a BGW

$$\begin{array}{cccc} \alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 \\ \alpha^{-2} & \alpha^{-1} & \alpha^{-4} & \alpha^{-3} \\ \hline \{\alpha^2 & \alpha^3 & \alpha^0 & \alpha^1\} \end{array}$$

Example of a BGW

$$
\begin{array}{cccc}
\alpha^4 & \alpha^4 & \alpha^4 & \alpha^4 \\
\alpha^{-2} & \alpha^{-1} & \alpha^{-4} & \alpha^{-3} \\
\hline
\{\alpha^2 & \alpha^3 & \alpha^0 & \alpha^1\}
\end{array}
$$

- Each element of $\mathbb{Z}_4$ appears exactly $\lambda/|G| = 4/4 = 1$ time.

The core of W(6,5)

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- Recall our weighing matrix $W(6,5)$:

$$W = \left[\begin{array}{c|ccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{array}\right]$$

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- The *core* of the matrix is:

$$C = \begin{bmatrix} 0 & 1 & - & - & 1 \\ 1 & 0 & 1 & - & - \\ - & 1 & 0 & 1 & - \\ - & - & 1 & 0 & 1 \\ 1 & - & - & 1 & 0 \end{bmatrix}$$

The core of W(6,5)

- Recall our weighing matrix $W(6,5)$:

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- The *core* of the matrix is:

$$C = \begin{bmatrix} 0 & 1 & - & - & 1 \\ 1 & 0 & 1 & - & - \\ - & 1 & 0 & 1 & - \\ - & - & 1 & 0 & 1 \\ 1 & - & - & 1 & 0 \end{bmatrix}$$

- Recall that the five rows of C provide an optimal $(5,4,4)_3$ code as $A_3(5,4,4) \leq 5 = M$.

An application

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- Using the weighing matrix $W(6,5)$ and O, the $OA(5,6)$ we can construct a $BGW(31,25,20)$ over $\mathbb{Z}_2$.

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- Using the weighing matrix $W(6,5)$ and O, the $OA(5,6)$ we can construct a $BGW(31,25,20)$ over $\mathbb{Z}_2$.
- Changing the entry i in O with the i-th row of C results in the *Derived* part of the $BGW(31,25,20)$ over $\mathbb{Z}_2$.
- Using $W(6,5)$ we compute $W \otimes (11111)$ which results in the *Residual* part of the following BGW.

A $BGW(31, 25, 20)$ over $\mathbb{Z}_2$

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$$
B = \left[\begin{array}{c|cccccccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - \\
0 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - \\
0 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 \\
1 & 0 & 1 & - & - & 1 & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - \\
1 & 0 & 1 & - & - & 1 & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - \\
1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 \\
1 & 0 & 1 & - & - & 1 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 \\
1 & 1 & 0 & 1 & - & - & 0 & 1 & - & - & 1 & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & 1 & - & - & 1 & 0 \\
1 & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & 1 & - & - & 1 & 0 & 0 & 1 & - & - & 1 \\
1 & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & 1 & - & - & 1 & 0 & 0 & 1 & - & - & 1 & 1 & 0 & 1 & - & - \\
1 & 1 & 0 & 1 & - & - & - & - & 1 & 0 & 1 & 1 & - & - & 1 & 0 & 0 & 1 & - & - & 1 & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & - \\
1 & 1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 & 0 & 1 & - & - & 1 & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 \\
1 & - & 1 & 0 & 1 & - & 0 & 1 & - & - & 1 & - & 1 & 0 & 1 & - & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & - & - & 1 & 0 & 1 \\
1 & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & - & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 \\
1 & - & 1 & 0 & 1 & - & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 & - & 1 & 0 & 1 & - \\
1 & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & - & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 & - & 1 & 0 & 1 & - & 1 & - & - & 1 & 0 \\
1 & - & 1 & 0 & 1 & - & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 & - & 1 & 0 & 1 & - & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - \\
1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 & 1 & 1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 & - & 1 & 0 & 1 & - \\
1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & 1 & 1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 & - & 1 & 0 & 1 & - & 0 & 1 & - & - & 1 \\
1 & - & - & 1 & 0 & 1 & 1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 & - & 1 & 0 & 1 & - & 0 & 1 & - & - & 1 & - & - & 1 & 0 & 1 \\
1 & - & - & 1 & 0 & 1 & 1 & - & - & 1 & 0 & - & 1 & 0 & 1 & - & 0 & 1 & - & - & 1 & - & - & 1 & 0 & 1 & 1 & 0 & 1 & - & - \\
1 & - & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & 0 & 1 & - & - & 1 & - & - & 1 & 0 & 1 & 1 & 0 & 1 & - & - & 1 & - & - & 1 & 0 \\
1 & 1 & - & - & 1 & 0 & 0 & 1 & - & - & 1 & 1 & - & - & 1 & 0 & - & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & - \\
1 & 1 & - & - & 1 & 0 & 1 & - & - & 1 & 0 & - & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & - & 0 & 1 & - & - & 1 \\
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1 & 1 & - & - & 1 & 0 & - & 1 & 0 & 1 & - & 1 & 0 & 1 & - & - & 0 & 1 & - & - & 1 & 1 & - & - & 1 & 0 & - & - & 1 & 0 & 1 \\
1 & 1 & - & - & 1 & 0 & 1 & 0 & 1 & - & - & 0 & 1 & - & - & 1 & 1 & - & - & 1 & 0 & - & - & 1 & 0 & 1 & - & 1 & 0 & 1 & -
\end{array}\right].
$$

An application of the $BGW(31, 25, 20)$ over $\mathbb{Z}_2$ to optimal ternary codes

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$$\mathcal{D} = \left[\begin{array}{*{30}{c}}
0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1 \\
0&1&-&-&1&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&- \\
0&1&-&-&1&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&- \\
0&1&-&-&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1 \\
0&1&-&-&1&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0&1&-&-&1&0 \\
1&0&1&-&-&0&1&-&-&1&1&0&1&-&-&-&1&0&1&-&-&-&1&0&1&1&-&-&1&0 \\
1&0&1&-&-&1&0&1&-&-&-&1&0&1&-&-&-&1&0&1&1&-&-&1&0&0&1&-&-&1 \\
1&0&1&-&-&-&1&0&1&-&-&-&1&0&1&1&-&-&1&0&0&1&-&-&1&1&0&1&-&- \\
1&0&1&-&-&-&-&1&0&1&1&-&-&1&0&0&1&-&-&1&1&0&1&-&-&-&1&0&1&- \\
1&0&1&-&-&1&-&-&1&0&0&1&-&-&1&1&0&1&-&-&-&1&0&1&-&-&-&1&0&1 \\
-&1&0&1&-&0&1&-&-&1&-&1&0&1&-&1&-&-&1&0&1&0&1&-&-&-&-&1&0&1 \\
-&1&0&1&-&-&1&0&1&-&1&-&-&1&0&1&0&1&-&-&-&-&1&0&1&0&1&-&-&1 \\
-&1&0&1&-&1&-&-&1&0&1&0&1&-&-&-&-&1&0&1&0&1&-&-&1&-&1&0&1&- \\
-&1&0&1&-&1&0&1&-&-&-&-&1&0&1&0&1&-&-&1&-&1&0&1&-&1&-&-&1&0 \\
-&1&0&1&-&-&-&1&0&1&0&1&-&-&1&-&1&0&1&-&1&-&-&1&0&1&0&1&-&- \\
-&-&1&0&1&0&1&-&-&1&-&-&1&0&1&1&0&1&-&-&1&-&-&1&0&-&1&0&1&- \\
-&-&1&0&1&-&-&1&0&1&1&0&1&-&-&1&-&-&1&0&-&1&0&1&-&0&1&-&-&1 \\
-&-&1&0&1&1&0&1&-&-&1&-&-&1&0&-&1&0&1&-&0&1&-&-&1&-&-&1&0&1 \\
-&-&1&0&1&1&-&-&1&0&-&1&0&1&-&0&1&-&-&1&-&-&1&0&1&1&0&1&-&- \\
-&-&1&0&1&-&1&0&1&-&0&1&-&-&1&-&-&1&0&1&1&0&1&-&-&1&-&-&1&0 \\
1&-&-&1&0&0&1&-&-&1&1&-&-&1&0&-&-&1&0&1&-&1&0&1&-&1&0&1&-&- \\
1&-&-&1&0&1&-&-&1&0&-&-&1&0&1&-&1&0&1&-&1&0&1&-&-&0&1&-&-&1 \\
1&-&-&1&0&-&-&1&0&1&-&1&0&1&-&1&0&1&-&-&0&1&-&-&1&1&-&-&1&0 \\
1&-&-&1&0&-&1&0&1&-&1&0&1&-&-&0&1&-&-&1&1&-&-&1&0&-&-&1&0&1 \\
1&-&-&1&0&1&0&1&-&-&0&1&-&-&1&1&-&-&1&0&-&-&1&0&1&-&1&0&1&-
\end{array}\right].$$

An application to ternary codes

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Example

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Proof.

Each row of $\mathcal{D}$ consists of 6 rows of C (recall C is the core of $W(6, 5)$).

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$$\begin{aligned} A_3(30,20,24) &\leq \left\lfloor \frac{nd(q-1)}{qw^2 - 2(q-1)nw + (q-1)nd} \right\rfloor \\ &= \left\lfloor \frac{2(30)(20)}{3(24)^2 - 2(2)(30)(24) + 2(30)(20)} \right\rfloor . \\ &= \left\lfloor \frac{1200}{48} \right\rfloor = 25 \end{aligned}$$

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Since $\mathcal{D}$ consists of 25 codewords, it follows that $A_3(30,20,24) \leq 25 = M$ and the constant weight code is optimal. □

Another application to ternary codes

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Example

Let B_{31} be the $BGW(31, 25, 20)$ over $\mathbb{Z}_2$. The rows of the matrix $\begin{bmatrix} B_{31} \\ -B_{31} \end{bmatrix}$ form an *optimal constant weight* $(31, 20, 25)_3$ *code*.

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Using the properties of $BGW(31, 25, 20)$ the Hamming distance of any two distinct rows of $\pm B_{31}$ is 20.

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Using the properties of $BGW(31, 25, 20)$ the Hamming distance of any two distinct rows of $\pm B_{31}$ is 20. Similarly, a row from B_{31} and a row from $-B_{31}$ will also have minimum Hamming distance 20.

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Using the properties of $BGW(31,25,20)$ the Hamming distance of any two distinct rows of $\pm B_{31}$ is 20. Similarly, a row from B_{31} and a row from $-B_{31}$ will also have minimum Hamming distance 20. We apply Johnson Bound (1) to obtain

$$A_3(31,20,25) \leq \left\lfloor \frac{n(q-1)}{w} A_q(n-1,d,w-1) \right\rfloor \leq \left\lfloor \frac{2(31)}{25} A_3(30,20,24) \right\rfloor$$
$$= \left\lfloor \frac{62}{25}(25) \right\rfloor = 62.$$

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Let B_{31} be the $BGW(31, 25, 20)$ over $\mathbb{Z}_2$. The rows of the matrix $\begin{bmatrix} B_{31} \\ -B_{31} \end{bmatrix}$ form an *optimal constant weight* $(31, 20, 25)_3$ *code*.

Proof.

Using the properties of $BGW(31, 25, 20)$ the Hamming distance of any two distinct rows of $\pm B_{31}$ is 20. Similarly, a row from B_{31} and a row from $-B_{31}$ will also have minimum Hamming distance 20. We apply Johnson Bound (1) to obtain

$$A_3(31, 20, 25) \leq \left\lfloor \frac{n(q-1)}{w} A_q(n-1, d, w-1) \right\rfloor \leq \left\lfloor \frac{2(31)}{25} A_3(30, 20, 24) \right\rfloor$$
$$= \left\lfloor \frac{62}{25}(25) \right\rfloor = 62.$$

As $B_{31} \cup -B_{31}$ consists of 62 codewords, it follows that $A_3(31, 20, 25) \leq 62 = M$ and the constant weight code is optimal. □

The new class of optimal ternary codes

Theorem

If p is an odd prime power and m is a positive integer, then

$$A_3\left(\frac{p^{m+1}-1}{p-1}, p^{m-1}\left(\frac{p+3}{2}\right), p^m\right) = 2\left(\frac{p^{m+1}-1}{p-1}\right).$$

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Example

We have seen the case for $p = 5$ and $m = 2$. For $m = 3$ we would recursively construct the matrix $B_{156} = BGW(156, 125, 100)$ over $\mathbb{Z}_2$.

The new class of optimal ternary codes

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The End!
Thank You!