Positive co-degree and unusual stability

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Co-degree for r-graphs

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For an r-graph H, the *minimum positive co-degree* of H is

$$\delta^+_{r-1}(H) = \min\{d_{r-1}(S) : S \subset V(H), |S| = r-1, d_{r-1}(S) > 0\}.$$

Forbidding r-graphs

Let F be a fixed r-graph. Let $\mathcal{S}_n$ be the set of n-vertex, F-free r graphs. We define

$$coex(n, F) := \max\{\delta_{r-1}(H) : H \in \mathcal{S}_n\}$$

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The study of $coex(n, F)$ is well established; $co^+ex(n, F)$ is a natural related question recently introduced.

One example

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H., Lemons, and Palmer showed that

$$co^+ex(n, K_4^-) = \lfloor n/3 \rfloor$$

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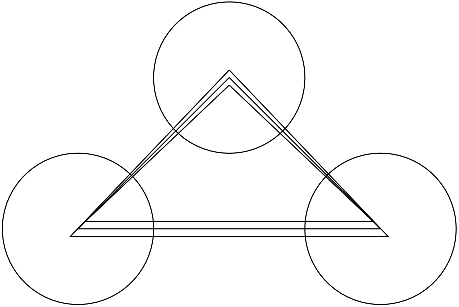
Extremal 3-graphs

Consider the balanced blow-up of a single 3-edge:



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It is also true that when $n \equiv 0 \mod 6$, the balanced n-vertex blow-up of H_6 has minimum positive co-degree $n/3$.

Two extremal constructions

Theorem (H.- Lemons- Palmer, 2022+)

$$co^{+}ex(n, K_4^-) = \lfloor n/3 \rfloor.$$

Moreover, suppose H achieves $co^{+}ex(n, K_4^-)$. If $n \equiv 3 \mod 6$, then H is the balanced blow-up of a 3-edge. If $n \equiv 0 \mod 6$, then H is either the balanced blow-up of a 3-edge or the balanced blow-up of H_6.

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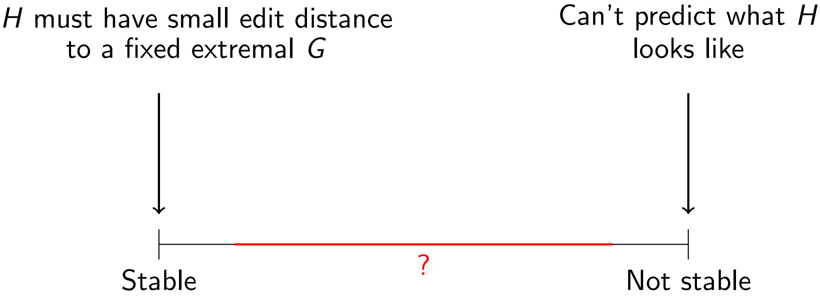
Is this behavior interesting?

The stability spectrum

H is near extremal



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If we can find t distinct constructions so that any near-extremal r-graph has small edit distance from one, we say our problem is *t-stable*

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t-stability results for $t > 1$ are less common. Liu and Mubayi recently found the first hypergraph Turán problem with 2-stability.

2-stability for K_4^-

Theorem (Garcia-H.)

Let H be an n-vertex, K_4^--free 3-graph with

$$\delta_2^+(H) = n/3 - o(n).$$

Then H has edit distance $o(n^3)$ from either the n-vertex balanced blow-up of a 3-edge or the n-vertex balanced blow-up of H_6.

Ingredients

- Identify a "special" vertex v of H and look at the link graph $L(v)$;
- Argue that, because v is special, we can transform $L(v)$ into one of two forms with few edits;
- Argue that, if $L(v)$ is of a good form, then we need only few edits to change to an extremal construction

Ingredients

Theorem (Frankl-Füredi, 1984)

Let H be an n-vertex 3-graph in which any four vertices span either 0 or 2 3-edges. Then H is either isomorphic to a blow-up of H_6, or to a 3-graph obtained by placing n points on the unit circle, with edges corresponding to triangles containing the origin.

The special vertex

Inspired by the Frankl-Füredi theorem, we will let $b(v)$ be the number of 4-sets of vertices which contain v and span exactly 1 edge.

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Because $\delta_2^+(H) > n/3 - o(n)$, it is quick to check that some v has $b(v) = o(n^4)$. This is our "special" vertex.

The link graph

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Certain subgraphs of $L(v)$ indicate bad sets containing v in H. Since v is special, we can indicate that we don't see too many of these "bad" subgraphs.

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We can also show that these edits maintain the "specialness" of v.

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Last step: edges not involving v also must go where we expect. (Similar work to showing that the link graph looks nice.)

Thanks for your attention!
Questions?