Bilinear multipliers in Orlicz spaces on Locally Compact Groups

Serap Öztop

İstanbul University
oztops@istanbul.edu.tr

Joint work with Alen Osançliol

CAHAS, Banff, June 2022

Orlicz Spaces

Definition (Young Function)

A nonzero function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is convex, $\Phi(0) = 0$, and $\lim_{x\to\infty} \Phi(x) = \infty$.

Definition (Complementary Young Function)

For a Young function Φ, the complementary (Young) function Ψ of Φ is

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\} \quad (y \geq 0).$$

- (Φ, Ψ) is called a complementary pair.
- We have the Young inequality

$$xy \leq \Phi(x) + \Psi(y) \quad (x, y \geq 0).$$

Let G be a locally compact abelian group with a fixed Haar measure ds.

Definition (Orlicz Space)

Given a Young function Φ, the **Orlicz Space** $L^\Phi(G)$ is defined to be

$$L^\Phi(G) = \left\{ f : G \to \mathbb{C} : \int_G \phi(\alpha|f|)ds < \infty \text{ for some } \alpha > 0 \right\}.$$

The Orlicz space $L^\Phi(G)$ is a Banach space under the following norms:

- Orlicz norm: $\|f\|_\Phi = \sup \left\{ \int_G |f(s)v(s)|ds : \|\Psi(|v|)\|_1 \le 1 \right\}$
- Luxemburg norm: $N_\Phi(f) = \inf \left\{ k > 0 : \int_G \Phi\left(\frac{|f(s)|}{k}\right) ds \le 1 \right\}.$

It is known that these two norms are equivalent with

$$N_\Phi(\cdot) \le \|\cdot\|_\Phi \le 2N_\Phi(\cdot)$$

A Young function Φ satisfies the Δ_2 condition (writing $\Phi \in \Delta_2$) if there exist $K > 0$ and $x_0 \geq 0$ such that $\Phi(2x) \leq K\Phi(x)$ for all $x \geq x_0$.
If $\Phi \in \Delta_2$, then:

(i) Both the step functions and $C_c(G)$ are dense in $L^\Phi(G)$;

(ii) $L^\Phi(G)^* = L^\Psi(G)$.

If, in addition, $\Psi \in \Delta_2$, then the Orlicz space $L^\Phi(G)$ is a reflexive Banach space.

(Generalized) Hölder's Inequality for Orlicz Spaces

For all $f \in L^\Phi(G)$ and $g \in L^\Psi(G)$, we have

$$\begin{aligned} \|fg\|_1 : &= \int_G |f(s)g(s)|ds \\ &\leq \min\{N_\Phi(f)\|g\|_\Psi, \|f\|_\Phi N_\Psi(g)\}. \end{aligned}$$

Example

For $1 \leq p < \infty$ and the Young function $\Phi(x) = \frac{x^p}{p}$, the space $L^{\Phi}(G)$ becomes the Lebesgue space $L^p(G)$ and the norm $\| \cdot \|_{\Phi}$ is equivalent to the classical norm $\| \cdot \|_p$.
If $p = 1$, then the complementary Young function of $\Phi(x) = x$ is

$$\Psi(y) = \begin{cases} 0 & \text{if } 0 \leq y \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

If $1 < p < \infty$, then the complementary Young function of $\Phi(x) = \frac{x^p}{p}$ is $\Psi(y) = \frac{y^q}{q}$, where q is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Bilinear Multiplier

Motivation

For a pair of functions $f, g : \mathbb{R} \to \mathbb{C}$ such that $\hat{f}$ and $\hat{g}$ are compactly supported and for any locally integrable function $m(\xi, \eta)$ defined on $\mathbb{R} \times \mathbb{R}$, one can consider the mapping

$$B_m(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

and ask about its boundedness on certain function spaces.

Definition (**O.Blasco, 2009**)

Let $1 \leq p_1, p_2 \leq \infty$ and $0 < p_3 \leq \infty$ and let locally integrable function $m(\xi, \eta)$ defined on $\mathbb{R} \times \mathbb{R}$. The function m is said to be a bilinear multiplier of type (p_1, p_2, p_3) if there exists $C > 0$ such that

$$\|B_m(f,g)\|_{p_3} \leqslant \|f\|_{p_1} \|g\|_{p_2}$$

for any $f, g \in S(\mathbb{R})$, which stands for the Schwartz class on $\mathbb{R}$.

That is, if B_m extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^{p_3}(\mathbb{R})$.
Denote by $BM_{(p_1,p_2,p_3)}(\mathbb{R})$ for the space of bilinear multipliers of type (p_1, p_2, p_3) and $\|m\|_{p_1,p_2,p_3} = \|B_m\|$.
Denote by $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ the space of measurable functions $M : \mathbb{R} \to \mathbb{C}$ such that $m(\xi,\eta) = M(\xi - \eta)$ belongs to $BM_{(p_1,p_2,p_3)}(\mathbb{R})$, that is,

$$B_M(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)M(\xi - \eta)e^{2\pi i<\xi+\eta,x>} d\xi d\eta$$

extends to a bounded bilinear map from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^{p_3}(\mathbb{R})$. We keep the notation $\|M\|_{p_1,p_2,p_3} = \|B_M\|$.

- O. Blasco has produce a method to get multipliers in $BM_{(p_1,p_2,p_3)}(\mathbb{R})$ from those in $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ and investigated some properties of these multiplier spaces.

Bilinear multipliers acting on other groups such as the torus $\mathbb{T}$ or the integers $\mathbb{Z}$ in place of $\mathbb{R}$ have also been studied. More recently, several results on bilinear multipliers acting on Orlicz spaces have been obtained.

O.Blasco and A. Osançliol, 2019

Let Φ_1, Φ_2 and Φ_3 be Young functions and let $L^{\Phi_1}(\mathbb{R}), L^{\Phi_2}(\mathbb{R})$ and $L^{\Phi_3}(\mathbb{R})$ be the corresponding Orlicz spaces. A locally integrable function m defined on $\mathbb{R} \times \mathbb{R}$ is said to be a bilinear multiplier of type (Φ_1, Φ_2, Φ_3) if there exists $C > 0$ such that

$$B_m(f,g)(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x} d\xi d\eta$$

satisfies

$$N_{\Phi_3}(B_m(f,g)) \leq C N_{\Phi_1}(f) N_{\Phi_2}(g)$$

for any $f, g \in \mathcal{S}(\mathbb{R})$. They investigated some properties of the spaces $BM_{(\Phi_1,\Phi_2,\Phi_3)}(\mathbb{R})$ and $\tilde{\mathcal{M}}_{(\Phi_1,\Phi_2,\Phi_3)}(\mathbb{R})$.

Similar results hold also on $\mathbb{R}^n$.

Our Goal:

- Use general locally compact abelian groups instead of special groups like $\mathbb{R}$, etc.
- Use Orlicz spaces $L^{\Phi}(G)$ on locally compact abelian groups.
- Replace the the Schwartz space $S(\mathbb{R})$ by the Feichtinger algebra $S_0(G)$.
- Give some sufficient conditions to define a bilinear multiplier on Orlicz space.

Technical Notes on LCA Groups

Let G be a locally compact abelian group.

Definition

Let $\mathbb{T}$ denote unit circle. A group homomorphism $\xi : G \to \mathbb{T}$ is called a character of G. The set $\hat{G}$ of all continuous characters of G is called the dual group of G.

- $\hat{G}$ is an abelian group under pointwise multiplication of functions.
- Under the compact-open topology $\hat{G}$ is a topological group. In fact, $\hat{G}$ becomes a locally compact abelian group.

Definition

Let $f \in L^1(G)$. The Fourier transform of $\mathcal{F}f$ of f is the function

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_G f(x)\overline{\langle \xi, x\rangle}dx \ , \ \xi \in \hat{G}.$$

Let $f \in L^1(G)$ with $\hat{f} \in L^1(\hat{G})$. The function f can be recovered from $\hat{f}$ by the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \int_{\widehat{G}} \hat{f}(\xi)\langle \xi, x\rangle d\xi,$$

a.e. $x \in G$. Next to the Fourier transform we need the following operators.

- Translation by $y \in G$: $\tau_y f(x) = f(x - y)$, $x \in G$.
- Modulation by $w \in \hat{G}$: $M_w f(x) =< w, x > f(x)$, $x \in G$.

All of these operators are well-defined, linear and bounded operators on Orlicz spaces.

Feichtinger Algebra(H.G. Feichtinger, 1981)

Recall the properties of Feichtinger's remarkable Segal algebra $S_0(G)$, i.e., translation invariant dense subalgebra of $L^1(G)$ under convolution which furthermore is continuously embedded into $L^1(G)$.

- $S_0(G)$ is the smallest Segal algebra in $L^1(G)$ that is closed under pointwise multiplication by characters and on which multiplication by any character is an isometry.
- Fourier transform induces an isomorphism $S_0(G) = S_0(\hat{G})$, where $\hat{G}$ is the dual group.
- $S_0(G)$ is the smallest Segal algebra in the Fourier algebra $A(G)$ that is translation invariant and on which translations are isometries.
- $S_0(G)$ is dense in $L^p(G)$ for $1 \leq p < \infty$.

Feichtinger Algebra in Orlicz Spaces

Let G be a locally compact abelian group. Denote $\Lambda_K(G)$ by

$$\Lambda_K(G) = \{f \in L^1(G) \mid supp(\hat{f}) \text{ is compact } \}.$$

Theorem (H. Reiter, 1968)

$\Lambda_K(G)$ is a dense subspace of $L^1(G)$.

Using the inclusions,

$$\Lambda_K(G) \subseteq S_0(G) \subseteq L^1(G) \cap L^\Phi(G) \subseteq L^\Phi(G). \tag{1}$$

we obtain the following results.

Theorem

If $\Phi \in \Delta_2$, $\Lambda_K(G)$ is a dense subspace of $L^\Phi(G)$.

Theorem

If $\Phi \in \Delta_2$, $S_0(G)$ is a dense subspace of $L^\Phi(G)$.

Bilinear Multipliers on $L^{\Phi}(G)$

Definition

Given three Young functions Φ_i for $i = 1, 2, 3$, a function $m \in L^{\infty}(\widehat{G} \times \widehat{G})$ is said to be a *bilinear multiplier* of type $(\Phi_1, \Phi_2; \Phi_3)$ if there exists a constant $C > 0$ such that

$$B_m(f,g)(x) = \int_{\widehat{G}} \int_{\widehat{G}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)\langle \xi + \eta, x\rangle d\xi d\eta$$

satisfies

$$N_{\Phi_3}(B_m(f,g)) \leq C N_{\Phi_1}(f) N_{\Phi_2}(g)$$

for any $f, g \in \mathcal{S}_0(G)$.

We write $\mathcal{BM}_{(\Phi_1,\Phi_2;\Phi_3)}(G)$ for the space of bilinear multipliers of type $(\Phi_1, \Phi_2; \Phi_3)$ with the norm $\|m\|_{(\Phi_1,\Phi_2;\Phi_3)} = \|B_m\|$. We denote by $\tilde{\mathcal{M}}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$ the space of locally integrable functions M defined on $\widehat{G}$ such that $m(\xi, \eta) = M(\xi - \eta) \in \mathcal{BM}_{(\Phi_1,\Phi_2;\Phi_3)}(G)$.

1. Note that both spaces are invariant under translation and modulation.

Theorem

Let Φ_1, Φ_2 and Φ_3 be Young functions such that

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq \Phi_3^{-1}(x), x \in \mathbb{R}.$$

If $m(\xi,\eta) = \hat{\mu}(\xi+\eta)$ a regular Borel measure μ on G, then $m \in \mathcal{BM}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$ and $\|m\|_{(\Phi_1,\Phi_2,\Phi_3)} \leq 2\|\mu\|$, where $\hat{\mu}(\xi) = \int_G \overline{\langle \xi, x \rangle} d\mu$, $\xi \in \hat{G}$.

Corollary

Let (Φ,Ψ) be a complementary pair of Young functions. If $m(\xi,\eta) = \hat{\mu}(\xi+\eta)$ where μ is a regular Borel measure on G then $m \in \mathcal{BM}_{(\Phi,\Psi,1)}$ and $\|m\|_{(\Phi,\Psi,1)} \leq 4\|\mu\|_1$.

Corollary

If $\frac{1}{p}+\frac{1}{q}=1$ for $1<p,q<\infty$ and $m(\xi,\eta)=\hat{\mu}(\xi+\eta)$ for a regular Borel measure μ on G, then $m \in \mathcal{BM}_{(p,q,1)}$ and $\|m\|_{(p,q,1)} \leq 4\|\mu\|_1$.

We can get a new bilinear multipliers from a given one.

Theorem

Let Φ_i for $i = 1, 2, 3$ be Young functions.

(a) If $\varphi \in L^1(\hat{G} \times \hat{G})$ and $m \in \mathcal{BM}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$, then $\varphi * m \in \mathcal{BM}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$ and $\|\varphi * m\|_{(\Phi_1,\Phi_2,\Phi_3)} \leq \|\varphi\|_1 \|m\|_{(\Phi_1,\Phi_2,\Phi_3)}$.

(b) If $\varphi \in L^1(G \times G)$ and $m \in \mathcal{BM}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$, then $\hat{\varphi} m \in \mathcal{BM}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$ and $\|\hat{\varphi} m\|_{(\Phi_1,\Phi_2,\Phi_3)} \leq \|\varphi\|_1 \|m\|_{(\Phi_1,\Phi_2,\Phi_3)}$.

Bilinear multipliers when $m(\xi, \eta) = M(\xi - \eta)$

Theorem

Let Φ_1, Φ_2 and Φ_3 be Young functions such that

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x) \leq x\Phi_3^{-1}(x), x \in \mathbb{R}.$$

If $M \in L^1(\widehat{G})$ then, $M \in \tilde{\mathcal{M}}_{(\Phi_1,\Phi_2,\Phi_3)}(G)$. Moreover

$$\|M\|_{(\Phi_1,\Phi_2,\Phi_3)} \leq 2\|M\|_1.$$

Corollary

Let (Φ, Ψ) be a complementary pair of Young functions. If $M \in L^1(\widehat{G})$, then $M \in \tilde{\mathcal{M}}_{(\Phi,\Psi,\infty)}(G)$. Moreover $\|M\|_{(\Phi,\Psi,\infty)} \leq 2\|M\|_1$.

Corollary

Let $p, q, r \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If $M \in L^1(\widehat{G})$, then $M \in \tilde{\mathcal{M}}_{(p,q,r)}(G)$ and $\|M\|_{(p,q,r)} \leq 2\|M\|_1$.

(1) If $\Phi(x) = x\ln(1+x)$, then $\Psi(x) \approx \cosh x - 1$.

(2) If $\Phi(x) = \cosh x - 1$, then $\Psi(x) \approx x\ln(1+x)$.

(3) If $\Phi(x) = e^x - x - 1$, then $\Psi(x) = (1+x)\ln(1+x) - x$.

(4) If $\Phi(x) = (1+x)\ln(1+x) - x$, then $\Psi(x) = e^x - x - 1$.

Note: For two Young Functions Ψ_1 and Ψ_2 we say $\Psi_1 \approx \Psi_2$ if $\exists 0 < a \leq b < \infty$ such that

$$\Psi_1(ax) \leq \Psi_2(x) \leq \Psi_1(bx) \qquad (x \geq 0).$$

References

- Blasco,O. *Notes on the Spaces of Bilinear Multipliers.* Rev. Unión Mat. Argentina **50** [2009], 20-34.
- Blasco, O., Osançlıol, A. *Notes on Bilinear Multipliers on Orlicz Spaces.* Math. Nachr. **292** [2019], 2522-2536.
- Feichtinger, H.G., *On a New Segal Algebra* Monatsh. Math.**92**, [1981], 269-289.
- Jacobsen, M.S., *On a (No Longer) New Segal Algebra: A Review of the Feichtinger Algebra*. J. Fourier Anal. Appl. **24** [2018], 1579-1660.
- Reiter, H.,*Classical Harmonic Analysis and Locally Compact Groups.*, Oxford University Press. [1968].
- Rao, M. M., Ren, Z.D., *Theory of Orlicz spaces.* CRM Press [1991].