The Affine Group of the plane and a new Continuous Wavelet Transform

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joint work with Keith Taylor

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Overview

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The map $V_\eta : \mathcal{H}_\pi \to L^2(G)$ is an isometry called the *continuous wavelet transform* associated to π and Equation (3) is the *reconstruction formula*.

The 1D CWT

The *affine group of the line* is

$$G_1 = \mathbb{R} \rtimes \mathbb{R}^* = \{[x, a] \mid x, a \in \mathbb{R}, a \neq 0\}$$

with group product $[x, a][y, b] = [x + ay, ab]$.

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G_1 has a square–integrable irreducible representation ρ that acts on $L^2(\mathbb{R})$ by

$$\rho[x, a]f(t) = |a|^{-1/2} f\left(\frac{t-x}{a}\right),$$

for all $t \in \mathbb{R}$, $f \in L^2(\mathbb{R})$, and $[x, a] \in G_1$.

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The classic continuous wavelet transform in one dimension arises from the fact that ρ is square–integrable.

Affine groups in two dimensions

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$$\widehat{\mathbb{R}^2} = \{\underline{\omega} = (\omega_1, \omega_2) : \omega_1, \omega_2 \in \mathbb{R}\}.$$

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Then $\widehat{\rho}[\underline{x}, A]\xi(\underline{\omega}) = |\det(A)|^{1/2} e^{2\pi i \underline{\omega x}} \xi(\underline{\omega} A)$, for $\underline{\omega} \in \widehat{\mathbb{R}^2}$ and for $\xi \in L^2(\widehat{\mathbb{R}^2})$.

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Theorem: (Bernier & Taylor, Führ)

Let H be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. The natural representation of $\mathbb{R}^n \rtimes H$ is square–integrable if and only if there exists an $\underline{\omega} \in \widehat{\mathbb{R}^n}$ such that $\underline{\omega}H$ is open and dense in $\widehat{\mathbb{R}^n}$ and the stabilizer $H_{\underline{\omega}}$ is compact.

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When $n = 2$ there are only a few examples where the conditions of this theorem apply.

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Example (3), with $\alpha = 1/2$, leads to the *Continuous Shearlet Transform*, which is especially useful for detecting edge singularities in images.

Are there any other useful groups that leads to a CWT ?

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My main project is to work out details of harmonic analysis of square-integrable functions on the group G_2 of all invertible affine transformations of $\mathbb{R}^2$.

To do this, we had to re-parametrize the 2×2 invertible matrices and express left invariant integration on G_2 in the new parameters. The results of our calculations,

Research group $G_2 = \mathbb{R}^2 \rtimes GL_2(\mathbb{R})$

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Haar measure of $\mathrm{GL}_2(\mathbb{R})$

$\mathrm{GL}_2(\mathbb{R})$ is a unimodular group and the Haar integral is given for $f \in C_c\big(\mathrm{GL}_2(\mathbb{R})\big)$,

$$\int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{da\, db\, dc\, dd}{(ad - bc)^2}$$

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Left Haar measure on G_2 is given by for $f \in C_c(G_2)$,

$$\int_{G_2} f \, d\mu_{G_2} = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{\mathbb{R}^2} f[\underline{y}, B] \frac{d\underline{y} \, dB}{|\det(B)|}$$

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We need to factorize $\mathrm{GL}_2(\mathbb{R})$ and we were not able to find a useful factorization in any paper or book. We get the idea of factorising $\mathrm{GL}_2(\mathbb{R})$ as $K_0 H_{(1,0)}$. This factorization is what makes some complicated calculations easier to do.

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Proposition

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, then A can be uniquely decomposed as $A = M_A C_A$, where

$$M_A = \begin{pmatrix} s & -t \\ t & s \end{pmatrix}, \text{ with } s = \frac{d(ad-bc)}{b^2+d^2}, t = \frac{-b(ad-bc)}{b^2+d^2},$$

and

$$C_A = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}, \text{ with } u = \frac{cd+ab}{(ad-bc)}, v = \frac{b^2+d^2}{(ad-bc)}.$$

Factorization of $\mathrm{GL}_2(\mathbb{R})$

The parametrization resulting from factoring $\mathrm{GL}_2(\mathbb{R})$ as $K_0\, H_{(1,0)}$ gives an alternate expression for the Haar integral. Haar integration on $\mathrm{GL}_2(\mathbb{R})$ is given by

$$\int_{\mathrm{GL}_2(\mathbb{R})} f\, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \right) \frac{ds\, dt\, du\, dv}{|v|(s^2 + t^2)}$$

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Theorem

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1. $K_0 \cap H_{(1,0)} = \{\mathrm{id}\}$
2. $\mathrm{GL}_2(\mathbb{R}) = K_0 H_{(1,0)} = \{MC : M \in K_0, C \in H_{(1,0)}\}$.

Haar measure of G_2 in the new parametrization

Note that we can now factor the group $G_2 = KH$, where

$$K = \left\{ \left[\underline{0}, \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \right] : s, t \in \mathbb{R}, s^2 + t^2 \neq 0 \right\}$$

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Let μ_{G_2}, μ_K, and μ_H denote the left Haar measures on G_2, K, and H, respectively. Then,

$$\int_{K_0} f \, d\mu_{K_0} = \int_{\mathbb{R}} \int_{\mathbb{R}} f \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \frac{ds\, dt}{s^2 + t^2},$$

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$$\int_K f \, d\mu_K = \int_{K_0} f[\underline{0}, M] \, d\mu_{K_0}(M) = \int_{\mathbb{R}} \int_{\mathbb{R}} f \left[\underline{0}, \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \right] \frac{ds\,dt}{s^2 + t^2},$$

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$$\int_{H} f \, d\mu_{H} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} f \left[\underline{x}, \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \right] \frac{d\underline{x}\,du\,dv}{|v|^3}$$

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Recall,

$$\int_{\mathrm{GL}_2(\mathbb{R})} f \, d\mu_{GL_2(\mathbb{R})} = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} f\left(\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \right) \frac{ds\, dt\, du\, dv}{|v|(s^2+t^2)}$$

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Thus, we can write

$$\int_{\mathrm{GL}_2(\mathbb{R})} f\, d\mu_{\mathrm{GL}_2(\mathbb{R})} = \int_{K_0}\int_{H_{(1,0)}} f(MC)\,|\det(C)|\, d\mu_{H_{(1,0)}}(C)\, d\mu_{K_0}(M)$$

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$$G_2 = \left\{ \left[\underline{0}, \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \right] \left[\underline{x}, \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \right] : \underline{x} \in \mathbb{R}^2, s, t, u, v \in \mathbb{R}, v \neq 0, s^2 + t^2 \neq 0 \right\}$$

$$= \left\{ \left[\begin{pmatrix} s & -t \\ t & s \end{pmatrix} \underline{x}, \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \right] : \underline{x} \in \mathbb{R}^2, s, t, u, v \in \mathbb{R}, v \neq 0, s^2 + t^2 \neq 0 \right\}$$

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$$\int_{G_2} f \, d\mu_{G_2} =$$

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π^1 an irreducible representation of $H_{(1,0)}$

Recall, $H_{(1,0)} = \left\{ \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} : u, v \in \mathbb{R}, v \neq 0 \right\}$.

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For $\begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix} \in H_{(1,0)}$ and $f \in L^2(\mathbb{R}^*)$,

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Well-known theorem

The left regular representation, $\lambda_{H_{(1,0)}}$, of $H_{(1,0)}$ is equivalent to a direct sum of infinitely many copies of π^1.

$\chi_{(1,0)} \otimes \pi^1$ an irreducible representation of H

Because $\chi_{(1,0)}$ is left fixed by $H_{(1,0)}$, we can combine $\chi_{(1,0)}$ with π^1 to make a representation of $H = \mathbb{R}^2 \rtimes H_{(1,0)}$.

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$$L^2\big(K, L^2(\mathbb{R}^*)\big) = \left\{ F : K \to L^2(\mathbb{R}^*) : \int_K \|F[\underline{0}, L]\|^2_{L^2(\mathbb{R}^*)} d\mu_K[\underline{0}, L] < \infty \right\}.$$

Representation $\sigma \sim ind_H^{G_2}(\chi_{(1,0)} \otimes \pi^1)$

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We found a way to clarify the meaning of this formula.

An important homeomorphism

Define a map $\gamma : \mathcal{O} = \widehat{\mathbb{R}^2} \setminus \{\underline{0}\} \to K_0$ by

$$\gamma(\omega_1, \omega_2) = \frac{1}{\omega_1^2 + \omega_2^2} \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}, \text{ for } (\omega_1, \omega_2) \in \mathcal{O}.$$

We can use γ to move σ to an equivalent representation acting on $L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$.

An important homeomorphism

Define a map $\gamma : \mathcal{O} = \widehat{\mathbb{R}^2} \setminus \{\underline{0}\} \to K_0$ by

$$\gamma(\omega_1, \omega_2) = \frac{1}{\omega_1^2 + \omega_2^2} \begin{pmatrix} \omega_1 & -\omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}, \text{ for } (\omega_1, \omega_2) \in \mathcal{O}.$$

We can use γ to move σ to an equivalent representation acting on

$$L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right).$$

Note that $L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$ is really just $L^2(\mathbb{R}^3)$, written in a convenient way.

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$$C^{-1}_{A^{-1}\gamma(\underline{\omega})} = \begin{pmatrix} 1 & 0 \\ u_{\underline{\omega},A} & v_{\underline{\omega},A} \end{pmatrix},$$

For some $u_{\underline{\omega},A}, v_{\underline{\omega},A} \in \mathbb{R}$.

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Calculations give

$$u_{\underline{\omega},A} = \frac{(ac+bd)(\omega_1^2 - \omega_2^2) - (a^2+b^2-c^2-d^2)\omega_1\omega_2}{(a\omega_1 + c\omega_2)^2 + (b\omega_1 + d\omega_2)^2}$$

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$$v_{\underline{\omega},A} = \frac{(ad - bc)(\omega_1^2 + \omega_2^2)}{(a\omega_1 + c\omega_2)^2 + (b\omega_1 + d\omega_2)^2}.$$

Representation σ_1

Define $U : L^2\big(K, L^2(\mathbb{R}^*)\big) \to L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$ by, for $F \in L^2\big(K, L^2(\mathbb{R}^*)\big)$ and $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$,

$$(UF)(\underline{\omega}, \omega_3) = \begin{cases} \frac{\big(F[\underline{0},\gamma(\underline{\omega})]\big)(\omega_3^{-1})}{\|\underline{\omega}\|\cdot|\omega_3|^{1/2}} & \text{for } \underline{\omega} \in \mathcal{O}, \omega_3 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

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$\sigma_1 \sim \sigma$ and

$$\big(\sigma_1[\underline{x}, A]\xi\big)(\underline{\omega}, \omega_3) = \frac{|\det(A)| \cdot \|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i(\underline{\omega x} + \omega_3 u_{\underline{\omega},A})} \xi\big(\underline{\omega}A, \omega_3 v_{\underline{\omega},A}\big),$$

for a.e. $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$ and all $\xi \in L^2\big(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\big)$.

Main Theorems

$$\left(\sigma_1[\underline{x}, A]\xi\right)(\underline{\omega}, \omega_3) = \frac{|\det(A)| \cdot \|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i(\underline{\omega x} + \omega_3 u_{\underline{\omega}, A})} \xi\left(\underline{\omega}A, \omega_3 v_{\underline{\omega}, A}\right),$$

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Theorem A:

As defined above, σ_1 is an irreducible representation of G_2.

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$$V_\psi\xi[\underline{x}, A] = \langle \xi, \sigma_1[\underline{x}, A]\psi \rangle_{L^2(\widehat{\mathbb{R}^2}\times\widehat{\mathbb{R}})},$$

for $[\underline{x}, A] \in G_2, \xi \in L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$, V_ψ is an isometry of $L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$ into $L^2(G_2)$.

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$$V_\psi \sigma_1[\underline{x}, A] = \lambda_{G_2}[\underline{x}, A] V_\psi, \text{ for } [\underline{x}, A] \in G_2.$$

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This shows σ_1 is equivalent to a subrepresentation of the left regular representation. Moreover, The left regular representation, λ_{G_2}, of G_2 is equivalent to a direct sum of infinitely many copies of σ_1.

Main Theorems

A function $\psi \in L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$ is called a σ_1-*wavelet* if

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For each $\underline{x} \in \mathbb{R}^2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$, define $\psi_{\underline{x},A}$ on $\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$ by

$$\psi_{\underline{x},A}(\underline{\omega}, \omega_3) = \frac{|\det(A)| \cdot \|\underline{\omega}\|}{\|\underline{\omega}A\|} e^{2\pi i(\underline{\omega}\underline{x} + \omega_3 u_{\underline{\omega},A})} \psi\left(\underline{\omega}A, \omega_3 v_{\underline{\omega},A}\right),$$

For a.e. $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$. Then $\psi_{\underline{x},A} \in L^2(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}})$.

For each $\xi \in L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$, let

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Theorem B:

The Duflo-Moore operator C_{σ_1} associated with σ_1 is given by, for any $\xi \in L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$, $C_{\sigma_1}\xi(\underline{\omega}, \omega_3) = \|\underline{\omega}\|^{-1}|\omega_3|^{-1/2}\xi(\underline{\omega}, \omega_3)$, for a.e. $(\underline{\omega}, \omega_3) \in \widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}$.

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Theorem C:

Let $\psi \in L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$ be a σ_1-wavelet. Then, for any $\xi \in L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right)$,

$$\xi = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{\mathbb{R}^2} V_\psi \xi[\underline{x}, A]\, \psi_{\underline{x},A} \frac{d\underline{x}\, d\mu_{\mathrm{GL}_2(\mathbb{R})}(A)}{|\det(A)|}, \quad \text{weakly in } L^2\left(\widehat{\mathbb{R}^2} \times \widehat{\mathbb{R}}\right).$$

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Thank you!