A Tannaka–Krein theorem for topological semigroups and approximations of characters

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Dedicated to Professor A. T.-M. Lau
with admiration and respect

Part I: Tannaka–Krein Theorem for Semigroups

Notation

- S: topological semigroup.

- $\widehat{S}$: Finite-dimensional, continuous, irreducible, unitary representations of S.

- $T(S) = \text{span}\,\{\pi_{ij} \colon \pi \in \widehat{S}\} \subset C^b(S)$. (Trig. Polyn.)

- If $\mathcal{F} \subset \widehat{S}$,

$$T_{\mathcal{F}}(S) = \text{span}\,\{\pi_{ij} \colon \pi \in \mathcal{F}\} \subset T(S).$$

Strongly almost periodic functions

- $SAP(S) = \overline{T(S)} \subset C^b(S)$.
- In general, $SAP(S) \subsetneq AP(S)$.
- For topological groups G:

$$SAP(G) = \overline{T(G)} = AP(G).$$

- $SAP(S)$ is translation invariant, unital C^*-subalgebra of $C^b(S)$, and it has a unique two-sided invariant mean M.

Tannaka–Krein theorem

- **Theorem** If G is a topological group and

$$\varphi\colon T(G) \longrightarrow \mathbb{C}$$

is a linear functional such that

$$\varphi(|h|^2) \geq 0 \qquad \text{for all } h \in T(G),$$

then $\varphi(h) \geq 0$ if $h \geq 0$. In particular, φ is continuous and can be extended to $\overline{T(G)} = AP(G)$.

- Since $T(G)$ is not complete, the result does not follow from standard results in C^*-algebras.

Tannaka–Krein theorem

- Tannaka (1939) proved a special case of the theorem where φ is multiplicative, and preserves complex-conjugation. He used his result in constructing a duality theory for compact topological groups.

- Krein (1941) proved the more general form stated above.

- Hewitt and Zuckerman (1950) used this theorem for the study of approximation properties of characters on groups.

Further developments

- Almost periodic functions on semigroups, and semigroup compactifications have been extensively studied by many authors: Lau, Dales, Strauss, Filali, Galindo, Spronk, Stokke, and many others.

- Characters on semigroups have also been studied by many. These studies are mainly concerned with extension properties of characters.

- A recent work influenced by Hewitt–Zuckerman's paper, is Grow and Hare (2004) paper which proves the existence of characters that approximate random choices of signs.

In this talk. . .

• We discuss an extension of Tannaka–Krein theorem for linear maps on $T_{\mathcal{F}}(S)$ with values in C^*-algebras.

• We also discuss applications to an approximation problem of character on semigroups.

Tannaka–Krein Theorem for semigroups

• Let $\mathcal{F} \subset \widehat{S}$ be a set of representations such that $T_{\mathcal{F}}(S)$ is a unital, conjugate-closed, subalgebra of $T(S)$.

Let A is a C^*-algebra and

$$\varphi\colon\ T_{\mathcal{F}}(S) \longrightarrow A$$

be a linear map such that

$$\varphi(|h|^2) \geq 0 \quad \text{for all } h \in T_{\mathcal{F}}(S).$$

Then $\varphi(h) \geq 0$ if $h \geq 0$.

In particular, φ is continuous and can be extended to a positive linear map on $\overline{T_{\mathcal{F}}(S)} \subset SAP(S)$.

C^*-algebras vs $\mathbb{C}$

- By identifying the C^*-algebra A with a closed subalgebra of $\mathcal{B}(H)$, positivity of $\varphi(h)$ can be expressed as

$$\langle \varphi(h)\xi, \xi \rangle \geq 0 \qquad (\xi \in H).$$

- So we can reduce the problem to linear functionals

$$\varphi\colon T_{\mathcal{F}}(S) \longrightarrow \mathbb{C}.$$

Groups vs Semigroups

- Since representations are unitary, in the group case we have

$$\pi(s^{-1}) = \pi(s)^* = [\overline{\pi_{ji}(s)}].$$

- So in the proof, the use of inverse elements can be avoided by replacing $\pi_{ij}(s^{-1})$ with $\overline{\pi_{ji}(s)}$.

Idea of the proof

- Following an idea of Bochner, the proof breaks into two cases:

Case I: Only finitely many $\varphi(\pi_{jk})$ are nonzero.

Case II: Infinitely many $\varphi(\pi_{jk})$ are nonzero.

- Case II is proved using Case I and a limiting process via the following technical result:

Technical result

- **Theorem** Let $\mathcal{F}_0 \subset \widehat{S}$ be a finite set. There exists a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $T^+(S)$ with the following properties.

(i) For all $n \geq 1$,

$$f_n = \sum_{\pi \in \mathcal{F}_n} d_\pi \lambda_{\pi,n} \operatorname{tr}(\pi),$$

where $\mathcal{F}_n \supset \mathcal{F}_0$ is finite, $\lambda_{\pi,n} \in \mathbb{C}$, $d_\pi = \dim H_\pi$.

(ii) If $\pi \in \mathcal{F}_0$, $1 \leq j, k \leq d_\pi$, then

$$\langle M, f_n \overline{\pi}_{jk} \rangle = \lambda_{\pi,n} \delta_{jk}.$$

(iii) For all $\pi \in \mathcal{F}_0$,

$$\lim_{n \to \infty} \lambda_{\pi,n} = 1.$$

Part II: Approximations of Characters on Semigroups

A group compactification of S

- S: topological semigroup

- $\mathrm{Hom}\,(S, \mathbb{T})$: all characters on S (continuous or not).

- Let H be a subgroup of $\mathrm{Hom}\,(S, \mathbb{T})$.

- For each $\theta \in H$, let $\mathbb{T}_\theta = \mathbb{T}$ be the circle group. We consider the semigroup homomorphism

$$\Psi\colon\, S \longrightarrow \prod_{\theta \in H} \mathbb{T}_\theta, \quad s \mapsto \Psi(s) = (\theta(s))_{\theta \in H},$$

and define

$$b_H S = \overline{\Psi(S)}.$$

Then $b_H S$ is a closed topological subsemigroup of the compact group $\prod_{\theta \in H} \mathbb{T}_\theta$, hence it is a compact topological group.

Approximation of character on semigroups

- **Theorem** If H is any subgroup of $\mathrm{Hom}\,(S, \mathbb{T})$, then

$$\mathrm{Hom}\,(H, \mathbb{T}) = b_H S.$$

- **Corollary**

If $\chi \in \mathrm{Hom}\,(H, \mathbb{T})$, and $\theta_1, \ldots, \theta_m$ are elements of H and $\epsilon > 0$, then there exists an element $s \in S$ such that

$$|\chi(\theta_j) - \theta_j(s)| < \epsilon \quad (j = 1, \ldots, m).$$

- This extends Hewitt–Zuckerman approximation results to semigroups.

Multidimensional Kronecker's theorem I

- **Theorem** If $h_1, \ldots, h_n \in \mathbb{R}$ are such that $\{1, h_1, \ldots, h_n\}$ is linearly independent over $\mathbb{Q}$, and if $b_1, \ldots, b_n$ are arbitrary real numbers, and N and ϵ are positive, then there is an integer $m > N$ such that

$$|\exp(2\pi i b_j) - \exp(2\pi i m h_j)| < \epsilon \qquad (j = 1, 2, \ldots, n).$$

- The theorem specifies that m can be chosen in any semigroup $S = \{N+1, N+2, \ldots\}$ of $\mathbb{R}$, where $N \in \mathbb{N}$.

Multidimensional Kronecker's theorem II

- **Theorem** Let $h_1, \ldots, h_n \in \mathbb{R}$ be such that $\{1, h_1, \ldots, h_n\}$ is linearly independent over $\mathbb{Q}$, let $b_1, \ldots, b_n \in \mathbb{R}$, and $\epsilon > 0$. If S is any semigroup of $(\mathbb{R}, +)$ which contains a nonzero rational number, then there exists some $x \in S$ such that

$$|\exp(2\pi i b_j) - \exp(2\pi i x h_j)| < \epsilon \qquad (j = 1, 2, \ldots, n).$$

- The result gives more concrete information on how x can be chosen in Kronecker's theorem. In particular, x can be chosen to be a multiple of any prime number.

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Thank you for your attention.