Cohomological obstructions to lifting properties for full group C*-algebras

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Based on joint work with A. Ioana and P. Spaas

A. Ioana, P. Spaas and M. Wiersma, *Cohomological obstructions to lifting properties for full group C*-algebras*. **Geom. Funct. Anal.** 30 (5) (2020), 1402–1438.

Lifting property

Conventions:

- C*-algebras are unital
- groups are discrete

Definition

A C*-algebra A has the *lifting property* (LP) if whenever $\varphi\colon A \to B/J$ is a ucp map where B is a C*-algebra and J is a closed two-sided ideal of B, there exists a ucp $\widetilde{\varphi}\colon A \to B$ so that the diagram

$$\begin{tikzcd} & B \arrow[d, two heads] \\ A \arrow[ur, dotted, "\widetilde{\varphi}"] \arrow[r, "\varphi"'] & B/J \end{tikzcd}$$

commutes.

Lifting property

Theorem (Choi-Effros 1976)

Nuclear C-algebras have the LP.*

Theorem (Kirchberg 1994)

$\mathrm{C}^*(\mathbb{F}_d)$ *has the LP for* $2 \leq d \leq \infty$.

Local lifting property

Definition (Kirchberg 1993)

A C*-algebra A has the *local lifting property* (LLP) if whenever $\varphi\colon A \to B/J$ is a ucp map where B is a C*-algebra and J is a closed two-sided ideal of B, and $E \subset A$ is a finite dimensional operator system, there exists a ucp map $\psi\colon E \to B$ so that

$$\begin{tikzcd} & B \arrow[d, two heads] \\ E \arrow[ur, dotted, "\psi"] \arrow[r, "\varphi|_E"'] & B/J \end{tikzcd}$$

commutes.

Local lifting property

Theorem (Pisier 1996)

Full free products of C-algebras with the LLP have the LLP.*

Theorem (Pisier-Junge 1995)

$\mathrm{B}(\mathcal{H})$ *does not have the LLP.*

- Extremely involved proof using operator spaces
- Valette (1997): Shorter proof using Ramanujan graphs
- Pisier (2006): Shorter proof that uses free probability
- Ioana-Spaas-W. (2020): Shorter proof that uses 2-cohomology of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$

A tale of two C*-algebras

Two important C*-algebras:

- $C^*(\mathbb{F}_\infty)$: Every separable C*-algebra is a quotient of $C^*(\mathbb{F}_\infty)$
- $B(\mathcal{H})$: Every separable C*-algebra embeds inside of $B(\mathcal{H})$

Theorem (Kirchberg 1993)

A C-algebra A has the weak expectation property (WEP) if and only if*

$$A \otimes_{\mathrm{min}} C^*(\mathbb{F}_\infty) = A \otimes_{\mathrm{max}} C^*(\mathbb{F}_\infty).$$

Theorem (Kirchberg 1993)

A C-algebra A has the LLP if and only if*

$$A \otimes_{\mathrm{min}} B(\mathcal{H}) = A \otimes_{\mathrm{max}} B(\mathcal{H}).$$

Connes embedding conjecture

Theorem (Kirchberg 1993)

The Connes embedding conjecture (CEC) has a positive solution if and only if LLP implies WEP.

Theorem (Ji-Natarajan-Vidick-Wright-Yuen 2020)

The CEC is false.

Open Problem (Ozawa 2003)

Is the LLP equivalent to the LP for separable C*-algebras?

Main Problem

Problem

Find groups whose full group C*-algebras do not have the (L)LP.

- Ozawa 2004: "Although it seems full group C*-algebras rarely have the LLP, there is no example of groups whose full C*-algebra is known to fail the LLP"
- Pisier 2016: "Problem: Find more examples of groups either with or without LLP"

Examples of groups with LLP

Example

If G is the subgroup of a free product of amenable groups, then $C^*(G)$ has the LLP.

No other examples of groups whose full C*-algebra has the LLP are known.

Examples of groups without the (L)LP

Theorem (Ozawa 2004)

There exists G such that $\mathrm{C}^*(G)$ does not have the LP.

Theorem (Thom 2010)

There exists groups whose full group C-algebra does not have the LLP.*

- Two explicit families of groups are given

Sneak peak of our results

Theorem (Ioana-Spaas-W. 2020)

$C^*(\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z}))$ *does not have the LLP.*

- Implies $SL(n,\mathbb{Z})$ does not have LLP for $n \geq 3$
- Fist example of residually finite G such that $C^*(G)$ does not have LLP

2-cohomology

Definition

A map $c\colon G \times G \to \mathbb{T}$ is a *2-cocycle* if

$$c(g,h)c(gh,k) = c(g,hk)c(h,k)$$

for all $g, h, k \in G$.

Example

If $b\colon G \to \mathbb{T}$ is an arbitrary function, then $c\colon G \times G \to \mathbb{T}$ defined by

$$c(g,h) = b(g)b(h)\overline{b(gh)}$$

for $g, h \in G$ is a 2-cocycle. If a 2-cocyle can be written in this form, it is called a *2-coboundary*.

Main Theorem: LLP

Theorem (Ioana-Spaas-W. 2020)

Suppose G is a countable group, $H \leq G$ has relative property (T) in G. If there exists 2-cocycles $c_n\colon G \times G \to \mathbb{T}$ so that

1. *$c_n|_H$ is not a coboundary for H for each $n \in \mathbb{N}$,*
2. *$c_n(g, h) \to 1$ for all $g, h \in G$, and*
3. *for every $n \in \mathbb{N}$, there is a map $\pi_n\colon G \to \mathcal{U}(\mathcal{H}_n)$ into the unitary operator on some finite dimensional Hilbert space $\mathcal{H}_n$ so that $\pi_n(g)\pi_n(h) = c_n(g, h)\pi_n(gh)$ for all $g, h \in G$.*

Then $\mathrm{C}^*(G)$ does not have the LLP.

Corollaries of Main Result: LLP

Corollary (Ioana-Spaas-W. 2020)

Suppose R is a finitely generated commutative ring with unity such that $\{2x : x \in R\}$ is infinite. Then $\mathrm{C}^*(R^2 \rtimes \mathrm{SL}(2, R))$ *does not have the LLP.*

Remark

This can be used to recover nearly all of Thom's examples of groups without the LLP.

Main Result: LP

Theorem (Ioana-Spaas-W. 2020)

Assume G is a countable group, $H \leq G$ has relative property (T) in G. Further suppose that there is a p.m.p. action $G \curvearrowright^{\sigma} (X, \mu)$ such that $\sigma|_H$ is ergodic, and 2-cocycles $c_n \in \mathrm{Z}^2(G, \mathrm{L}^0(X, \mathbb{T}))$ so that

1. *$c_n|_H$ is not a coboundary for H for each $n \in \mathbb{N}$,*
2. *$\lim_{n \to \infty} \|c_n(g, h) - 1\|_{\mathrm{L}^2} = 0$, for every $g, h \in G$.*

Then $\mathrm{C}^*(G)$ does not have the LP.

Groups without the LP

Corollary (Ioana-Spaas-W. 2020)

Let G be a group with property (T). If either $\mathrm{H}^2(G, \mathbb{R})$ *or* $\mathrm{H}^2(G, \mathbb{Z}G)$ *is nontrivial, then* $\mathrm{C}^*(G)$ *does not have the LP.*

Example

Let G be a simple Lie group with trivial centre, infinite cyclic fundamental group and property (T). If $\Gamma \leq G$ is a lattice inside of G, then $\mathrm{C}^*(\Gamma)$ does not have the LP.

- $\mathrm{C}^*(\mathrm{Sp}(2n, \mathbb{Z}))$ does not have the LP for $n \geq 2$.

Groups without the LP

Theorem (Ioana-Spaas-W. 2020; Pisier 2020)

If G is a non-finitely presentable group with property (T), then $C^*(G)$ does not have the LP.

Question

Does every infinite property (T) group fail to have the (L)LP?

Thank you!