Constructing discrete frames from continuous wavelet transforms

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Overview

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Definition (Discrete Frame)

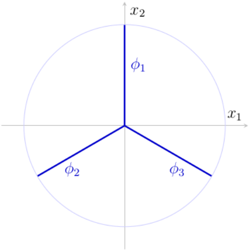
A set of vectors $\{\phi_i\}_{i=1}^{\infty}$ is a *discrete frame* for a Hilbert space $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ which depend exclusively on $\mathcal{H}$ and $\{\phi_i\}_{i=1}^{\infty}$ such that

$$A\,\|f\|_{\mathcal{H}}^2 \leq \sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq B\,\|f\|_{\mathcal{H}}^2 \qquad \forall f \in \mathcal{H}.$$

Example: Mercedes-Benz Frame for $\mathbb{R}^2$.

$$\phi_1 = \begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}, \qquad \phi_3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix}.$$

$$x = \sum_{i=1}^{3} \langle x, \phi_i \rangle \phi_i, \qquad \forall x \in \mathbb{R}^2.$$



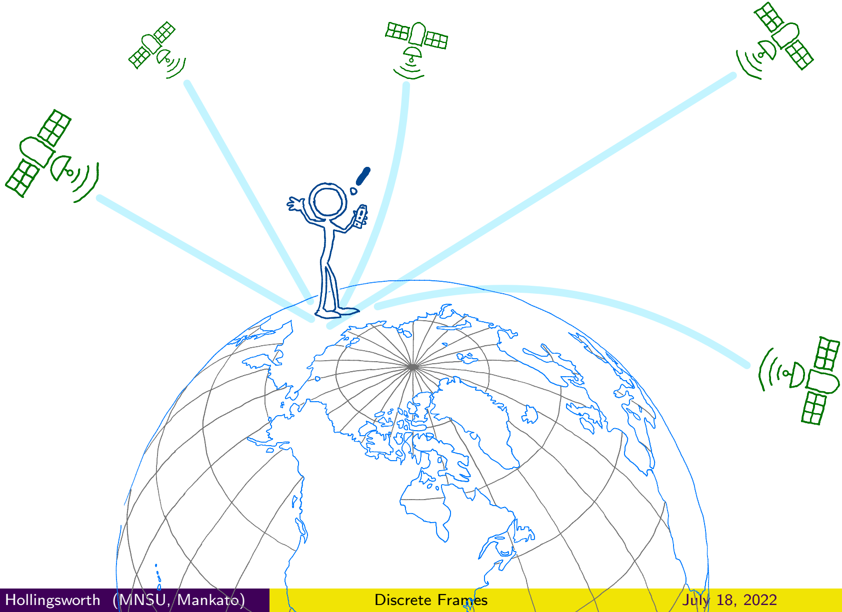
Frames vs. Bases

Recall:

$$A \|f\|_{\mathcal{H}}^2 \leq \sum_{i=1}^{\infty} |\langle f, \phi_i \rangle|^2 \leq B \|f\|_{\mathcal{H}}^2 \qquad \forall f \in \mathcal{H}.$$

- Frames replace linear independence with a stability condition.
  - Frame condition number $= \frac{B}{A}$
- If the frame is not a basis, the representation naturally contains some redundancy. (Error can live in the kernel of the reconstruction operator)
- The vectors are not necessary related. This makes implementation difficult.

An Example



Two Important Frame Constructions

Let $\psi \in L^2(\mathcal{H})$ be a fixed function.
$\psi^{a,\xi}$ denotes *translation* by a and *modulation* by ξ.
$\psi_{a,h}$ denotes *translation* by a and *dilation* by h.

$$\underbrace{\left\{ \psi^{a,\xi} \right\}_{(a,\xi)\in X_1}}_{\text{Gabor Frame}} \qquad \underbrace{\left\{ \psi_{a,h} \right\}_{(a,h)\in X_2}}_{\text{Wavelet Frame}}$$

Example: For $\psi \in L^2(\mathbb{R})$, we have

- Gabor frames: $\psi^{a,\xi}(x) = e^{2\pi i\xi}\,\psi(-a+x)$.
- Wavelet frames: $\psi_{a,h}(x) = \frac{1}{\sqrt{|h|}}\,\psi\left(\frac{-a+x}{h}\right)$

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Generalization to Continuous Frames

Definition

A *continuous frame* for a separable Hilbert space $\mathcal{H}$ is a collection of vectors $\{\phi_x\}_{x\in X}$ with X a locally compact Hausdorff space equipped with a positive measure μ satisfying

$$A\|f\|_{\mathcal{H}}^2 \leq \int_X |\langle f, \phi_x\rangle|^2 \, d\mu(x) \leq B\|f\|_{\mathcal{H}}^2 \qquad \forall f \in \mathcal{H}$$

for some positive real numbers A and B. Notice that when μ is the counting measure on a countable space X, this agrees with the previous definition of a discrete frame.

Key Ideas Coming Up

- Given a *square-integrable representation* of a group G, we can find a *wavelet*.
- Given a *wavelet*, there exists an isometry into $L^2(G)$ known as the *continuous wavelet transform*.
- The *continuous wavelet transform* can be used to find a *continuous frame*.
- Given this *continuous frame*, can we find a sampling set (discretization) to construct a discrete frame?

Remark: We need a 'Goldilocks zone' of sampling.

Continuous Unitary Group Representation

Let $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators on a Hilbert space $\mathcal{H}$ with operator composition.

Let G be a group.
$\pi : G \to \mathcal{U}(\mathcal{H})$ is called a *unitary representation* if π is a group homomorphism.

π is a *continuous unitary representation* if the map $\pi_{\xi,\eta} : G \to \mathbb{C}$ defined $\pi_{\xi,\eta}(x) = \langle \xi, \pi(x)\eta \rangle_{\mathcal{H}}$ is continuous for every $\xi, \eta \in \mathcal{H}$.

An *irreducible* representation π is called *square integrable* if $\exists \xi, \eta \in \mathcal{H} \setminus \{0\}$ such that

$$\int_G |\pi_{\xi,\eta}(x)|^2 \, dx < \infty.$$

Continuous Wavelet Transform (Dufflo and Moore, 1976)

For a square integrable representation π, there exists $\psi \in \mathcal{H}$ such that:

$$\begin{aligned} V_\psi : &\mathcal{H} \to L^2(G) \\ &\xi \mapsto \langle \xi, \pi(\cdot)\psi \rangle \end{aligned}$$

is an isometry.

We call ψ a wavelet, and V_ψ a continuous wavelet transform (CWT).

How CWT leads to a continuous frame.

As V_ψ is an isometry, we have

$$\begin{aligned}\langle f, g \rangle_{L^2(A)} &= \langle V_\psi f, V_\psi g \rangle_{L^2(G_n)} \\ &= \int_{G_n} \langle f, \psi_{x,h} \rangle_{L^2(A)} \langle \psi_{x,h}, g \rangle_{L^2(A)} \, d[x,h] \\ &= \left\langle \int_{G_n} \langle f, \psi_{x,h} \rangle \psi_{x,h} \, d[x,h], g \right\rangle_{L^2(A)}.\end{aligned}$$

In particular, we have

$$f = \int_{G_n} \langle f, \rho[x,h]\psi \rangle \, \rho[x,h]\psi \, d[x,h],$$

in the WOT. So the CWT leads to a continuous (Parseval) tight frame!

WOT: Weak Operator Topology **CWT:** Continuous Wavelet Transform

Our Group

Let $G_n = \mathrm{M}_n(\mathbb{R}) \rtimes \mathrm{GL}_n(\mathbb{R})$ be the group of affine transformations, which is a locally compact group with group product defined as

$$[x_1, h_1][x_2, h_2] = [x_1 + h_1 x_2, h_1 h_2].$$

- We think of $A = \mathrm{M}_n(\mathbb{R})$ as an abelian group.
- We think of $\mathrm{GL}_n(\mathbb{R})$ as acting on A.

(Some) Previous Work on Discretization Problem

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David Bernier and Keith F. Taylor.
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SIAM J. Math. Anal., 27(2):594–608, 1996.

Mahya Ghandehari, Aizhan Syzdykova, and Keith F. Taylor.
(2013) A four dimensional continuous wavelet transform.
In volume 603 of *Contemp. Math.*, pages 123–136. Amer. Math. Soc.

Key Ideas of Bernier and Taylor

- In general, start with the group $G_n = \mathrm{M}_n(\mathbb{R}) \rtimes H \subseteq \mathrm{GL}_n(\mathbb{R})$
- Define $\rho : A \rtimes H \to \mathcal{H}(L^2(A))$.

$$\rho[x,h] : L^2(A) \to L^2(A) \text{ for } [x,h] \in A \rtimes H \text{ by}$$

$$\rho[x,h]\, g(y) = |\det(h)|^{-1/2} g(h^{-1}(-x+y)), \quad \forall y \in A.$$

- Let $\pi(\cdot) = \mathcal{P}\rho(\cdot)\mathcal{P}^{-1}$, then $\pi : A \rtimes H \to \mathcal{H}(L^2(\widehat{A}))$ is an equivalent representation on the dual.
- If π or ρ is a square-integrable representation, then $\exists\, g \in L^2(A)$ such that g is a wavelet.
- Use a discretization ('Tiling System') of Fourier space to discretize the CWT to construct a discrete frame.

Remark: We have defined everything in the context of the affine group, but these techniques generalize to many other semidirect product groups. In particular, we use induced representations for semidirect product groups.

Our Method

We build on the work of Bernier, Ghandehari, Syzdykova, and Taylor.

Definition (Tiling System)

Let P be a countable subset of $\mathrm{GL}_n(\mathbb{R})$, and F be an open relatively compact subset of $\mathrm{GL}_n(\mathbb{R})$. The pair (F, P) is called a *tiling system* for $\mathrm{GL}_n(\mathbb{R})$ if the following two conditions are satisfied:

(i) $\lambda_{\mathrm{GL}_n}\left(\overline{F} \cdot p \bigcap \overline{F} \cdot q\right) = 0$ for every distinct pair $p, q \in P$,

(ii) $\lambda_{\mathrm{GL}_n}\left(\mathrm{GL}_n(\mathbb{R}) \setminus \bigcup_{p \in P} \overline{F} \cdot p\right) = 0$,

where λ_{GL_n} denotes the left Haar measure of $\mathrm{GL}_n(\mathbb{R})$.

Take $\overline{F} \subset F_{\mathrm{o}} \subset R$, with R a hypercube with ONB J and F_{o} an open set.

Define $M := \sup_{p \in P} \left| \left\{ k \in P \;\middle|\; \overline{F} \cdot p \cap F_{\mathrm{o}} \cdot k \neq \emptyset \right\} \right|$.

ONB: Orthonormal Basis

Theorem (Main Result [M.Ghandehari, K.H., 2021] J Math Anal Appl)

Let (F, P) *be a tiling system for* $\mathrm{GL}_n(\mathbb{R})$*. Let* $g \in L^2(\mathrm{M}_n(\mathbb{R}))$ *be such that*

$$\mathbb{1}_{\overline{F}} \leq \widehat{g} \leq \mathbb{1}_{F_\mathrm{o}}.$$

Then the collection of vectors $\left\{ \rho[\lambda, p]^{-1} g \;\middle|\; (\lambda, p) \in J \times P \right\}$ *is a discrete frame in* $L^2(\mathrm{M}_n(\mathbb{R}))$ *satisfying*

$$|R| \, \|f\|_2^2 \leq \sum_{k \in P} \sum_{\gamma \in J} \left| \langle f, \rho[\gamma, k]^{-1} g \rangle_{L^2(\mathrm{M}_n(\mathbb{R}))} \right|^2 \leq M|R| \, \|f\|_2^2.$$

Note: R, F_o, *and* M *as defined in the previous slide.*

A Simplified Proof (of Main Result) – Setup

Our Representations

Our group consists of the *affine transformations* on $\mathbb{R}^{n^2}$.

$$\rho : (\mathrm{M}_n(\mathbb{R}) \rtimes \mathrm{GL}_n(\mathbb{R})) \to \mathcal{U}(L^2(\mathrm{M}_n(\mathbb{R}))),$$
$$\rho[x, h]g(y) = |\det(h)|^{-n/2} g(h^{-1}(-x + y))$$

$$\pi : (\mathrm{M}_n(\mathbb{R}) \rtimes \mathrm{GL}_n(\mathbb{R})) \to \mathcal{U}(L^2(\widehat{\mathrm{M}_n(\mathbb{R})})),$$
$$\pi[x, h]\xi(\chi) = |\det(h)|^{n/2}\, \overline{\chi(x)}\, \xi(h^{-1} \cdot \chi)$$

Assumptions

$$F := [0,1]^{n^2} \sim \mathbb{T}^{n^2}, \qquad \widehat{g} := \mathbb{1}_F, \qquad \widehat{A} := \widehat{\mathrm{M}_n(\mathbb{R})} = \dot{\bigcup_{k\in P}} F\cdot k,\ P \text{ discrete.}$$

Discretization

$$\|f\|^2_{L^2(A)} = \|\widehat{f}\|^2_{L^2(\widehat{A})} = \int_{\widehat{A}} |\widehat{f}(\chi_c)|^2\, dc = \sum_{k\in P} \int_{F\cdot k} |\widehat{f}(\chi_c)|^2\, dc$$

$$= \sum_{k\in P} |\det k|^n \int_F |\widehat{f}(\chi_{bk})|^2\, db$$

$$\beta_k : \mathbb{T}^{n^2} \to \mathbb{C}$$

$$\boxed{\beta_k(b) := \widehat{f}(\chi_{bk})\overline{\widehat{g}(\chi_b)}}$$

$$= \sum_{k\in P} |\det k|^n \|\beta_k\|^2_{L^2(\mathbb{T}^{n^2})}$$

$$= \sum_{k\in P} |\det k|^n \|\mathcal{F}\beta_k\|^2_{\ell^2(\mathbb{Z}^{n^2})}$$

$$= \sum_{k\in P} |\det k|^n \sum_{j\in\mathbb{Z}^{n^2}} |(\mathcal{F}\beta_k)(j)|^2$$

Discretization (Continued)

$$\begin{aligned}
\|f\|^2 &= \sum_{k \in P} |\det k|^n \sum_{j \in \mathbb{Z}^{n^2}} |(\mathcal{F}\beta_k)(j)|^2 \\
&= \sum_{k \in P} |\det k|^n \sum_{j \in \mathbb{Z}^{n^2}} \left| \int_F \overline{\chi_b(j)} \widehat{f}(\chi_{bk}) \overline{\widehat{g}(\chi_b)} \, db \right|^2 \\
&= \sum_{k \in P} \sum_{j \in \mathbb{Z}^{n^2}} \left| \int_F |\det k|^{\frac{n}{2}} \overline{\chi_b(j)} \widehat{f}(\chi_{bk}) \overline{\widehat{g}(\chi_b)} \, db \right|^2 \\
&= \sum_{k \in P} \sum_{j \in \mathbb{Z}^{n^2}} \left| \int_F \pi[j,k] \widehat{f}(\chi_b) \overline{\widehat{g}(\chi_b)} \, db \right|^2 \\
&= \sum_{k \in P} \sum_{j \in \mathbb{Z}^{n^2}} \left| \langle \pi[j,k] \widehat{f}, \widehat{g} \rangle \right|^2 \\
&= \sum_{k \in P} \sum_{j \in \mathbb{Z}^{n^2}} \left| \langle f, \rho[j,k]^{-1} g \rangle \right|^2
\end{aligned}$$

That's great! But ...

how do we construct a tiling system for $\mathrm{GL}_n(\mathbb{R})$ exist?

Our Tiling System

Define the set F and discrete set P as

$$F = \left\{ sk \begin{pmatrix} w_1 & w_1 y_{1,2} & w_1 y_{1,3} & \cdots & w_1 y_{1,n-1} & w_1 y_{1,n} \\ & w_2 & w_2 y_{2,3} & \cdots & w_2 y_{2,n-1} & w_2 y_{2,n} \\ & & w_3 & \cdots & w_3 y_{3,n-1} & w_3 y_{3,n} \\ & & & \ddots & & \vdots \\ & & & & w_{n-1} & w_{n-1} y_{n-1,n} \\ & & & & & \prod_{i=1}^{n-1} w_i^{-1} \end{pmatrix} \middle| \begin{array}{r} s, w_i \in [1,2), \\ y_{i,j} \in [0,1), \\ k \in \mathrm{O}_n \end{array} \right\},$$

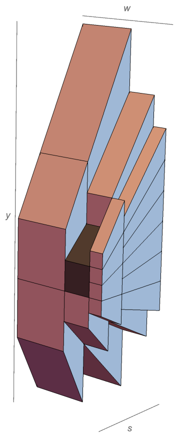
$$P = \left\{ 2^{\lambda} \begin{pmatrix} 2^{\kappa_1} & 2^{\kappa_2}\mu_{1,2} & 2^{\kappa_3}\mu_{1,3} & \cdots & 2^{\kappa_{n-1}}\mu_{1,n-1} & 2^{\kappa_n}\mu_{1,n} \\ & 2^{\kappa_2} & 2^{\kappa_3}\mu_{2,3} & \cdots & 2^{\kappa_{n-1}}\mu_{2,n-1} & 2^{\kappa_n}\mu_{2,n} \\ & & 2^{\kappa_3} & \cdots & 2^{\kappa_{n-1}}\mu_{3,n-1} & 2^{\kappa_n}\mu_{3,n} \\ & & & \ddots & & \vdots \\ & & & & 2^{\kappa_{n-1}} & 2^{\kappa_n}\mu_{n-1,n} \\ & & & & & 2^{\kappa_n} \end{pmatrix} \middle| \begin{array}{r} \lambda, \kappa_i, \mu_{i,j} \in \mathbb{Z}, \\ \kappa_n = -\sum_{i=1}^{n-1} \kappa_i \end{array} \right\}.$$

Our Tiling System

Define the set F and discrete set P as

$$F = \left\{ sk \begin{pmatrix} w & wy \\ & w^{-1} \end{pmatrix} \;\middle|\; \begin{matrix} s, w \in [1,2), \\ y \in [0,1), \\ k \in \mathrm{O}_n \end{matrix} \right\},$$

$$P = \left\{ 2^{\lambda} \begin{pmatrix} 2^{\kappa} & 2^{-\kappa}\mu \\ & 2^{-\kappa} \end{pmatrix} \;\middle|\; \lambda, \kappa, \mu \in \mathbb{Z}, \right\}.$$



Theorem ([M.Ghandehari, K.H., 2021] J Math Anal Appl)

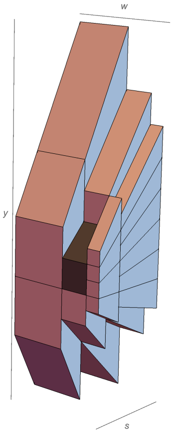
For F and P as in the previous slide, we have

1. $\lambda_{\mathrm{GL}_n}(\overline{F} \cdot p \cap \overline{F} \cdot q) = 0$ *for every distinct pair* $p, q \in P$,
2. $\lambda_{\mathrm{GL}_n}(\mathrm{GL}_n(\mathbb{R}) \setminus \bigcup\{\overline{F} \cdot p \ : \ p \in P\}) = 0$.

That is, they satisfy the definition of a tiling system. By the main result, we have

$$|R| \, \|f\|_2^2 \leq \sum_{k\in P} \sum_{\gamma\in J} \left|\langle f, \rho[\gamma, k]^{-1} g \rangle_{L^2(\mathrm{M}_n(\mathbb{R}))}\right|^2 \leq M|R| \, \|f\|_2^2.$$

Therefore our frame condition number is $\frac{|R|M}{|R|} = M$.



Frame Stability

Proposition [M.Ghandehari, K.H., 2021] (J Math Anal Appl)

For F, P, and $F_{\rm o}$ as before, and $0 < \epsilon \leq \frac{1}{2}$, we have

$$M = \sup_{p \in P} \left| \left\{ k \in P \;\middle|\; \overline{F} \cdot p \cap F_{\rm o} \cdot k \neq \emptyset \right\} \right| \leq 3^n 6^{\frac{n(n-1)}{2}}.$$

Corollary: For $n = 2$, $M \leq 36$.

Huge Improvement: The previous construction by Ghandehari, Syzdykova, and Taylor has a frame condition number of about 1700, and did not provide a construction for $n > 2$.

Remark: Tiling systems also generalize beyond the affine group.

Definition (Iwasawa Decomposition)

Let G be a semisimple Lie group, then the *Iwasawa decomposition* of G is

$$G = KAN,$$

where K, A and N are closed subgroups, K is compact, A is abelian, and the map

$$(k, a, n) \mapsto kan$$

is an analytic diffeomorphism.

For our matrix groups, this corresponds to the well-known QR-decomposition, with R further decomposed by LU-decomposition.

Work in Preparation

Theorem ([K.H. 2022+])

By showing the overlap of tiles is bounded by

$$\lambda_{\mathrm{GL}_n}(E_{p,p'}) \leq c_n \left[\frac{(2+\epsilon)^{2n-2} - 2^{2n-2}}{2n-2} \right],$$

for some constant c_n depending on the dimension, one can find classes of signals for which

$$\frac{B}{A} = \frac{|R| + f(\epsilon)}{|R|}$$

with $f(\epsilon) \to 0$ as $\epsilon \to 0$.

Our goal now is to find more natural descriptions of this set of functions.

Work in Preparation

Let H^s (for $s \in \mathbb{Z}^{\geq 0}$) denote the Sobolev space

$$H^s(\mathrm{M}_n(\mathbb{R})) := \left\{ f \in L^2(\mathrm{M}_n(\mathbb{R})) \;\middle|\; \mathcal{F}^{-1}\left[(1+|2\pi\xi|^2)^{s/2}\mathcal{F}f\right] \in L^2(\mathrm{M}_n(\mathbb{R})) \right\}.$$

Theorem ([S.Cogar, K.H., 2022+])

Let (F, P) be a tiling system for $\mathrm{GL}_n(\mathbb{R})$, with R, $F_\circ$, and J as before. Let $g \in H^s(\mathrm{M}_n(\mathbb{R}))$ be such that $\mathbb{1}_{\overline{F}} \leq \widehat{g} \leq \mathbb{1}_{F_\circ}$. Then the collection

$$\left\{ \rho[\lambda, p]^{-1} g \;\middle|\; (\lambda, p) \in J \times P \right\}$$

is a discrete frame for $H^s(\mathrm{M}_n(\mathbb{R}))$ if and only if $s = 0$.

In the future, we hope to provide a modified method to construct sampling sets in these spaces as well.

Current Work

Raja Milad and Keith F. Taylor.
Harmonic analysis on the affine group of the plane.

Raja Milad and Keith F. Taylor.
A Two Plus One Dimensional Continuous Wavelet Transform.

Let G be the group of invertible affine transformations of $\mathbb{R}^2$.

Theorem ([Milad, Taylor 2022+](Preprint))

Let $\psi \in L^2(\mathbb{R}^2 \times \mathbb{R})$ be a wavelet. Then, for any $f \in L^2(\mathbb{R}^2 \times \mathbb{R})$,

$$f = \int_{\mathrm{GL}_2(\mathbb{R})} \int_{\mathbb{R}^2} V_\psi f[\underline{x}, A]\, \psi_{\underline{x},A} \, \frac{d\underline{x}\, d\mu_{\mathrm{GL}_2(\mathbb{R})}(A)}{|\det(A)|}, \quad \textit{weakly in } L^2.$$

Working jointly with K.F. Taylor and M. Ghandehari we have begun work on discretizing this new CWT.

Thank you for your attention! Any questions?

Mahya Ghandehari and Kris Hollingsworth.
Discrete Frames For $L^2(\mathrm{M}_n(\mathbb{R}))$ Arising From Tiling Systems On $\mathrm{GL}_n(\mathbb{R})$.
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Mahya Ghandehari, Aizhan Syzdykova, and Keith F. Taylor.
A four dimensional continuous wavelet transform.
In *Commutative and noncommutative harmonic analysis and applications*, volume 603 of *Contemporary Mathematics*, pages 123–136. AMS, Providence, RI, 2013.

- Kris Hollingsworth.
Improved Frame Bounds on $L^2(\mathrm{M}_n(\mathbb{R}))$ for Subclasses of Signals.
In Preparation. 2022+.

- Samuel Cogar and Kris Hollingsworth.
Banach Frames for Sobolev Spaces Arising From Tiling Systems On $\mathrm{GL}_n(\mathbb{R})$.
In Preparation. 2022+.