Amenability Gaps for Central Fourier Algebras of Finite Groups

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Banach Algebra Amenability

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Theorem [Johnson]

The group algebra $L^1(G)$ is amenable if and only if G is an amenable group, in which case $AM(L^1(G)) = 1$.

Theorem [5, Johnson 1994]

Let G be a finite group, denote the irreducible characters on G by $\mathrm{Irr}(G)$, and let $A(G)$ be the Fourier algebra of G. Then

$$AM(A(G)) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^3$$

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This particular bound is sharp because $AM(A(D_4)) = \frac{3}{2}$.

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A finite group G is abelian if and only if $AM(ZL^1(G)) < \frac{7}{4}$. In this case $AM(ZL^1(G)) = 1$.

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This bound is also sharp, because $AM(ZL^1(D_4)) = \frac{7}{4}$.

The Central Fourier Algebra

For a compact group G denote the central Fourier algebra of G by

$$ZA(G) = A(G) \cap ZL^1(G)$$

where the norm is the $A(G)$ norm. If we restrict to finite groups then $ZA(G)$ and $ZL^1(G)$ are both equal to the class functions on G, albeit with different norms and multiplication.

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Importantly, the above gap is not necessarily sharp. The smallest known value for $AM(ZA(G))$ is $\frac{7}{4}$, and just like with $AM(ZL^1(G))$ it is achieved at D_4.

Theorem [2] and [3]

Let G be a finite group. Then

$$AM(ZL^1(G)) = \frac{1}{|G|^2} \sum_{C,C' \in \mathrm{Conj}(G)} |C||C'| \left| \sum_{\chi \in \mathrm{Irr}(G)} d_\chi^2 \chi_\pi(C) \overline{\chi_\pi(C')} \right|$$

and

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Calculations in GAP show that of the 851 non-abelian groups with order less than 100, there are 678 groups with $AM(ZL^1(G)) = AM(ZA(G))$. Interestingly, the first group of odd order that doesn't satisfy this has order 567.

Structure of Sum

What kind of values can $AM(ZA(G))$ achieve? Recall that

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Fact

Because irreducible characters have values in the algebraic integers, we know that $\left| \sum_{C \in \mathrm{Conj}(G)} |C|^2 \chi(C)\overline{\chi'(C)} \right| \in \mathbb{Z}$

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However, it turns out that taking the complex multitude is unnecessary, as the inner quantity is always an integer.

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Idea of Proof

- Use Clifford theory to create a partition of $\mathrm{Irr}(G)$ based on $\mathrm{Irr}(Z(G))$.
- Simplify the sum based on this partition.
- Use Galois theory to show that what remains is a rational algebraic integer, hence an integer.

Two Character Degrees and Two Conjugacy Classes

Theorem [1, Alaghmandan, Choi, Samei, 2014]

Let G be a non-abelian finite group such that every non-linear irreducible character has degree m. Then

$$AM(ZL^1(G)) = 1 + 2(m^2 - 1)\left(1 - \frac{1}{|G| \cdot |G'|} \sum_{C \in \mathrm{Conj}(G)} |C|^2\right)$$

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Theorem [S.]

Let G be a non-abelian finite group where all non-central conjugacy classes have size k. Then

$$AM(ZA(G)) = 2k - 1 + 2(1-k) \cdot \frac{|Z(G)|}{|G|^2} \cdot \left(\sum_{\chi \in \mathrm{Irr}(G)} d_\chi^4\right)$$

Two Character Degrees and Two Conjugacy Classes

Example

Let p be a prime. A finite group G is called p-extraspecial if

- $|Z(G)| = p$
- $G/Z(G)$ is non-trivial elementary abelian p-group

If the above is satisfied then $|G| = p^{2n+1}$, and G has both two character degrees and two conjugacy class sizes. Both the formulas for $AM(ZL^1(G))$ and $AM(ZA(G))$ apply and yield the same result, namely that

$$AM(ZL^1(G)) = AM(ZA(G)) = 1 + 2\left(1 - \frac{1}{p^{2n}}\right)\left(1 - \frac{1}{p}\right).$$

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Question

Does $AM(ZL^1(G)) = AM(ZA(G))$ hold for all finite groups with two character degrees and two conjugacy class sizes?

Hereditary Properties

$A(G)$ and $ZL^1(G)$

Both $AM(A(G))$ and $AM(ZL^1(G))$ possess nice hereditary properties:

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Example

If $G = C_8 \rtimes (C_2 \times C_2)$ and $N = D_8$ is identified as a normal subgroup of G, then $AM(ZA(G)) = 2.59375$ and $AMZA(N) = 2.6875$, so $AM(ZA(G)) < AMZA(N)$.

Property Q Groups

Definition

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Example

There is a group of order 192 $N \cong C_2$ in G such that $G/N \cong \text{SmallGroup}(96, 204)$, then $AM(ZA(G)) = 13.4921875$ and $AMZA(G/N) = 15.53125$, so G does not have property Q.

AM(ZA(G)) Gap Bound

Question

Is it true that a finite group G is abelian if and only if $AM(ZA(G)) < \frac{7}{4}$?

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Examples

The $\frac{7}{4}$ gap holds for the following classes of groups:

- All groups with order less than 384 (via GAP computations)
- Frobenius Groups with abelian factor and kernel
- Extraspecial $p-$groups
- Groups with property Q
- Any group G with $AM(ZA(G)) = AM(ZL^1(G))$

That's it, folks!

Thank you for attending my talk :)

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