An analogue of Slepian vectors on Boolean Hypercubes

BIRS Workshop on Multitaper Spectral Analysis

June 25, 2022

joint work with Jeff Hogan

Overview

1. Review: Thomson's multitaper method
2. Dyadic processes (motivation, no actual results here)
3. Thomson's method for dyadic processes requires *dyadic* optimizers of spatio–spectral limiting (SSL)
4. SSL on Hypercube ($\mathbb{Z}_2^N$) graphs: definitions
5. Identification and computation of eigenvectors of SSL on $\mathbb{Z}_2^N$

Thomson's multitaper method

Thomson [1982]: estimate power spectrum of a (stationary ergodic, Gaussian) process from N equally spaced samples of an instance by averaging K *tapered* periodograms.

$\{x(0), \ldots, x(N-1)\}$: N-contiguous sample observation

Cramér representation: $x(n) = \int_{-1/2}^{1/2} e^{2\pi i v[n-(N-1)/2]}\, dZ(v)$,

dZ: zero mean, orthogonal increments;

S: true spectrum of X

$S(f)\, df = E\{|dZ(f)|^2\}$.

Tapers: Slepian DPSS's (Fourier coefficients of DPSWFs)

DPSSs $v_n^{(k)}$ satisfy $\sum_{m=0}^{N-1} \frac{\sin 2\pi W(n-m)}{\pi(n-m)} v_m^{(k)} = \lambda_k v_n^{(k)}$

Spectrum estimate $\bar{S}(f_0)$: average of tapered eigenspectra

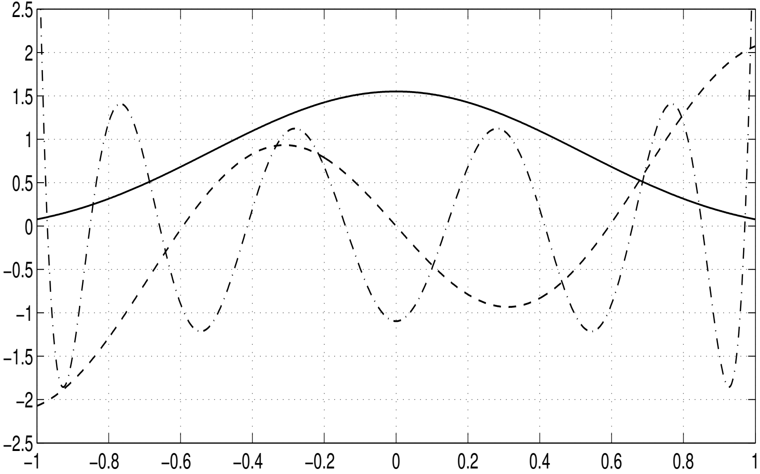


Figure: *(Continuous) prolates* φ_n, $n = 0, 3, 10$, $c = \pi T\Omega/2 = 5$

Dyadic processes

$X = \{X_n : n = 0, 1, 2, \dots\}$ is *dyadic stationary* if
$B(n, m) = \mathrm{cov}(X_n, X_m) = E(X_n X_m)$ depends only on $n \oplus m$
Dyadic representation: $n = \sum \epsilon_k(n) 2^k$,
$n \oplus m = \sum [(\epsilon_k(n) + \epsilon_k(m)) \bmod 2]\, 2^k$
Walsh functions $W(n, x)$ define the dyadic Fourier transform.
Hadamard–Walsh Fourier transform of $x(0), \dots, x(M-1)$ is
$(Hx)(\lambda) = \frac{1}{\sqrt{M}} \sum_{t=0}^{M-1} X(t) W(t, \lambda)$.
Dyadic stationary processes admit a spectral representation:
$X_n = \int_0^1 W(n, x)\, dZ(x)$
dZ: orthogonal increments; $E[|dZ(x)|^2] = dF(x)$ with
$B(\tau) = \int_0^1 W(\tau, x)\, dF(x)$.
F is called the *dyadic spectral distribution function* of X.
Morettin [1981, SIAM Review] Walsh spectrum estimation based on averaged Walsh periodograms of temporal slices.

Dyadic processes originally regarded as defined on $[0, 1]$

Interest in dyadic processes waned in late 1980s

Stoffer [JASA, 1991]: reviewed use in analysis of *categorical* data

Observed problem with insistence on concept of *dyadic time*

More appropriate for study of processes indexed by (limits of) $\mathbb{Z}_2^N$?

Graph Setting

Stationary Graph Processes and Spectral Estimation: Marques et al., 2017, IEEE Trans. Sig. Process.

Signals on Graphs: Uncertainty Principle and Sampling, Tsitsvero et al. , 2016, IEEE Trans. Sig. Process. ("prolates")

Finite version of Slepian sequences for $\mathbb{Z}_2^N$

$\mathbb{Z}_2^N = \{(\epsilon_1, \dots, \epsilon_N) : \epsilon_k \in \{0,1\}\}$
Q_K: truncation to K-Hamming nbhd of zero:
$\{(\epsilon_1, \dots, \epsilon_N) : \sum \epsilon_k \leq K\}$
$\mathbb{Z}_2^N$ has an isomorphic Fourier dual group
P_K: bandlimiting, $P_K = HQ_K H^T$
Fix $0 < K < N$. $P = P_K$, $Q = Q_K$.
BSV (Boolean Slepian Vectors) φ are eigenvectors of PQ:
$PQ\varphi = \lambda\varphi$ for suitable $\lambda > 0$

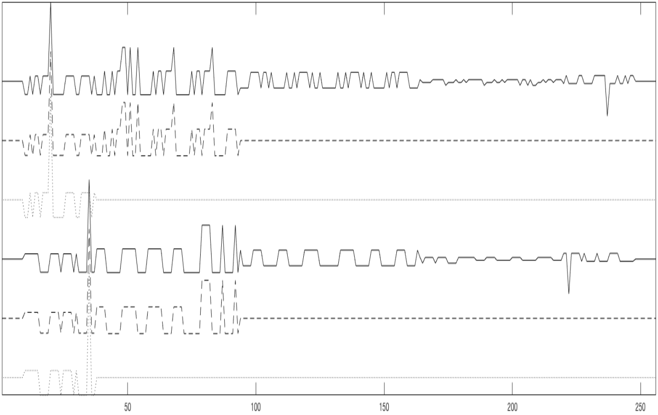


Figure: Eigenvectors of PQ on $\mathbb{Z}_2^N$, $N = 8$, $K = 3$, $r = 2$.
Dotted curves: two different elements g of $\mathcal{W}_r$
Dashed curves: corresponding eigenvectors f of QP
Solid curves: Eigenvector Hf of PQ for eigenvector f of QP

Comparison of properties

Properties in Common (with PSWFs in $L^2(\mathbb{R})$)

PSWF setting	Property	Hypercube (BSV) setting	Property
$\widehat{Q\varphi}_n = \pm\alpha i\sqrt{\lambda_n} v_n$	Trunc. Fourier eigenprop.	$Hv = \pm\sqrt{\lambda} Qv$	✓
$\lambda_n = \lVert Q\varphi_n \rVert^2$	Concentrations	$\lambda = \lVert Qv \rVert^2$	✓
$\int_{-1}^{1} \varphi_n(t)\,\varphi_m(t)\,dt = \delta_{nm}$	Double orthogonality	$\langle Qv, Qw \rangle = 0,\ v \neq w$	✓
$\mathrm{span}\,\varphi_n$ dense in $L^2[-1, 1]$	Local completeness	$\mathrm{span}\,\{v\} = \mathrm{range}\,Q$	✓
$\sum \lambda_k \lvert U_k \rvert^2 = \mathrm{const}$	Spectral accumulation	$\sum \lambda \lvert Hv \rvert^2 = \dim(K)$	✓

Differences

$\lambda_n > \lambda_{n+1}$	Simple eigenvalues	$\binom{N}{k} - \binom{N}{k-1}$	high multiplicity
$\lambda_n \approx 1,\ n < 2\Omega T$	$2\Omega T$-Theorem		✗
$\frac{d}{dt}(1 - t^2)\,\frac{d}{dt} - c^2 t^2$	Commuting differential op	$D(\alpha I - T^2)D + \beta T^2$, $D = HTH$	*Almost*

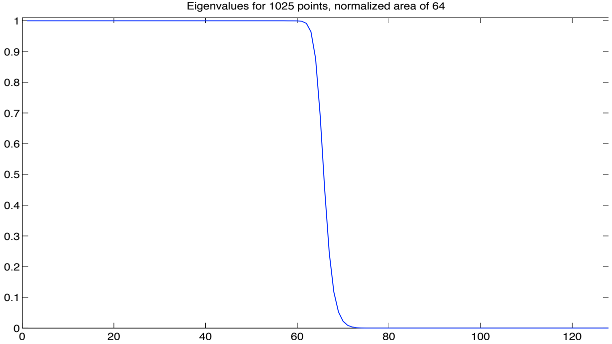


Figure: Eigenvalues of PQ for 1025 point DFT, $2NW \approx 64$

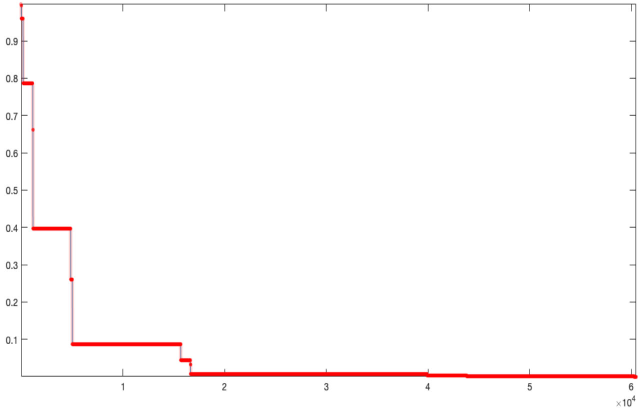


Figure: Eigenvalues of PQ, for Boolean FT on $\mathbb{Z}_2^{20}$, $K = 6$, with multiplicity, (60460). Corresponding case on $\mathbb{Z}_{2^{20}}$ would have about 3486 eigenvalues larger than $1/2$

GOAL: eigen-decomposition of PQ on $\mathcal{B}_N$

Outline

Geometry of $\mathcal{B}_N$

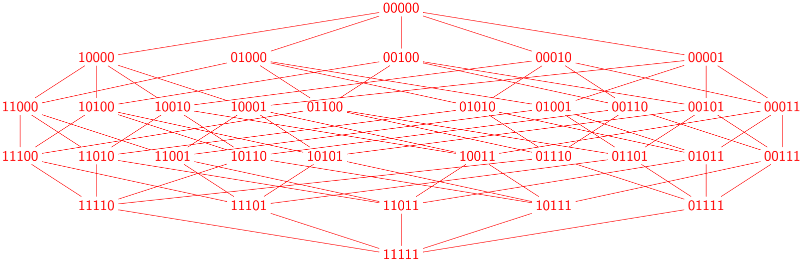
$D(\alpha I - T^2)D + \beta T^2$ almost commutes with PQ

Adjacency invariant spaces on which $D(\alpha I - T^2)D + \beta T^2$ acts as a tridiagonal matrix

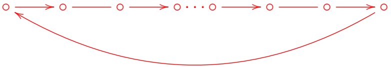
Basis of eigenvectors of BDO

Numerical method to compute eigenvectors of QP

Boolean cubes $\mathcal{B}_N$: $N = 5$



VS



Some conventions for $\mathcal{B}_N$

$v = (\epsilon_1, \ldots, \epsilon_N) \in \mathbb{Z}_2^N$
$S = \{i : \epsilon_i = 1\} \subset \{1, \ldots, N\}$
$v = v_S$ or "$v \sim S$"
Adjacency: $A_{RS} = 1$ if $R \Delta S$ is a singleton

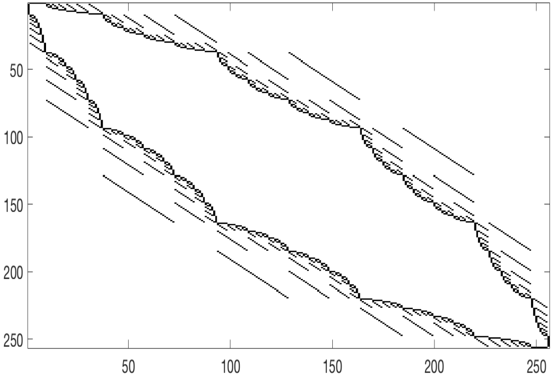


Figure: Adjacency matrix for $N = 8$ in dyadic lexicographic order.

Fourier transform H

The graph Fourier transform on $\mathcal{B}_N$ is the same as the group Fourier transform on $\mathbb{Z}_2^N$.
It is represented by a *Walsh–Hadamard matrix* H.

Lemma (Boolean Fourier transform)

Let $H_S(R) = 2^{-N/2}(-1)^{|R \cap S|}$ *and* $L = NI - A$ *(Laplacian of* $\mathcal{B}_N$*).*
Then H_S *is an eigenvector of* L *with eigenvalue* $2|S|$*.*

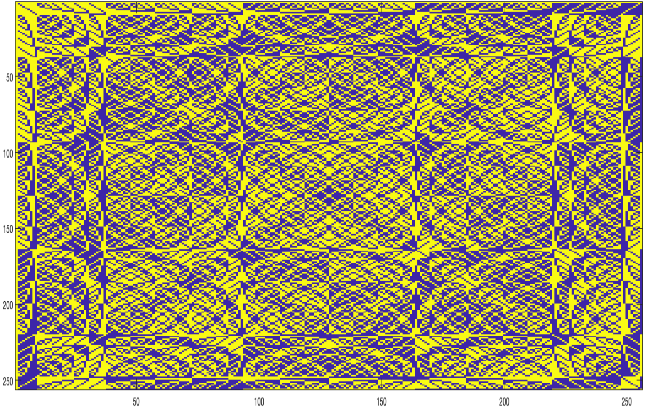


Figure: Hadamard (Fourier) matrix, $N = 8$ in dyadic lexicographic order.

Spatial and spectral limiting on $\mathcal{B}_N$

Space-limiting matrix $Q = Q_K$: $Q_{R,S} = \begin{cases} 1, & R = S \ \& \ |S| \leq K \\ 0, & \text{else} \end{cases}$

Spectrum-limiting matrix $P = P_K$ by $P = HQH$

Results and approach

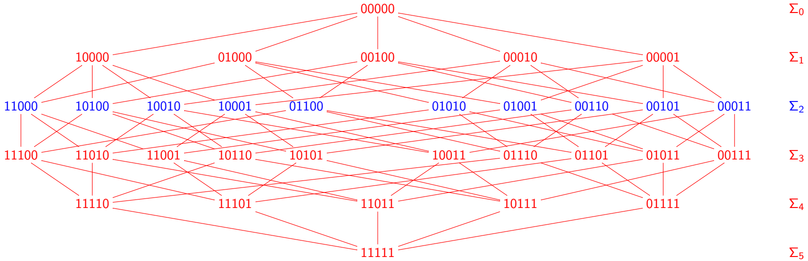
Results: identify eigenvectors of spatio–spectral limiting PQ

Approach:

- Work in *spectral domain*: $QP = HPQH$
- Identify salient *invariant subspaces* of QP
- Subspaces *factor*
- Reduce to *small matrix problem* on *radial* factor
- Eigenvectors of small matrix determine eigenvectors of QP
- *Numerical computation* via almost commuting operator and power method with a weight

Hamming spheres

Σ_r: Hamming sphere of radius r: vertices with r one-bits



Eigenspaces of SSL on $\mathcal{B}_N$: Adjacency-invariant spaces

A: adjacency matrix of $\mathcal{B}_N$ (dyadic lexicographic order)
$A = A_+ + A_-$: $A_- = A_+^T$; A_+: lower triangular
A_+ maps data on Σ_r to data on Σ_{r+1}: *outer adjacency*
A_- maps data on Σ_r to data on Σ_{r-1}: *inner adjacency*

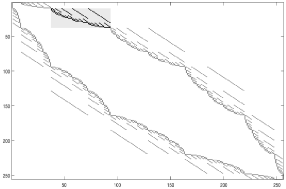


Figure: Highlighted: A_-, $\Sigma_3 \to \Sigma_2$

$\mathcal{W}_r$: the orthogonal complement of $A_+\ell^2(\Sigma_{r-1})$ inside $\ell^2(\Sigma_r)$.

$$\ell^2(\Sigma_r) = A_+\ell^2(\Sigma_{r-1}) \oplus \mathcal{W}_r$$

Theorem (Multiplier theorem)

Let $g \in \mathcal{W}_r$ and k such that $k \leq N - 2r$. Then

$$A_- A_+^{k+1} g = (k+1)(N-2r-k) A_+^k g \equiv m(r,k) A_+^k g$$

Adjacency invariant subspaces

$\mathcal{V}_r =: \operatorname{span}\{A_+^k g : g \in \mathcal{W}_r, k = 0, \ldots, N-2r\} \simeq \mathcal{W}_r \otimes \mathbb{R}^{N-2r+1}$

Lemma

A_+ and A_- map $\mathcal{V}_r$ to itself.

Idea: Fix $\mathcal{W}_r$ coordinate. A_+ *acts as right shift of coefficients:*
$(c_0 + c_1 A_+ + \ldots)g \mapsto (c_0 A_+ + c_1 A_+^2 + \ldots)g$
By *multiplier theorem*, A_- *acts as multiplicative left shift:*
$(c_0 + c_1 A_+ + \ldots)g \mapsto (c_1 m(r,0) + c_2 m(r,1) A_+ + \ldots)g$

Corollary

$A = A_+ + A_-$ maps $\mathcal{V}_r$ to itself. Polynomials $p(A)$ preserve $\mathcal{V}_r$.

Proposition

The spectrum-limiting operator $P = P_K$ can be expressed as a polynomial $p(A)$ of degree N.

Proof.

$$p_k = \prod_{j=0, j\neq k}^{N} \frac{x-(N-2j)}{2(j-k)}; \qquad p(x) = \sum_{k=0}^{K} p_k$$

Then $P = p(A)$ as verified on Hadamard basis. □

Coefficient matrices on $\mathcal{V}_r$: $M^P_{(r)} = p(M_A)$

Matrices M_{A_+}, M_{A_-} of A_+, A_- on $\mathbb{R}^{N-2r+1}$:

$$M_{A_+} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \; M_{A_-} = \begin{pmatrix} 0 & m(r,0) & 0 & \cdots & 0 \\ 0 & 0 & m(r,1) & 0 & \cdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & \cdots & 0 & m(r, K+1-r) \\ 0 & & \cdots & & 0 \end{pmatrix}$$

$$M_A = M_{A_+} + M_{A_-}$$

Matrix of $M^P_{(r)}$ of P by substituting M_A for A in $P = p(A)$

Matrix of $M^{QP}_{(r)}$ of QP by truncating $M^P_{(r)}$ to principal minor

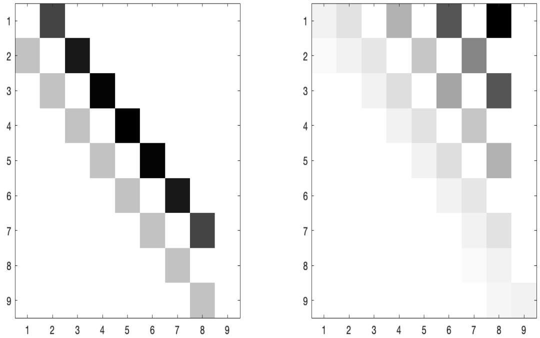


Figure: Matrices M_A and M^P , $N = 9$, $K = 4$, $r = 1$. (log scale)

Corollary

(i) Eigenvectors of coefficient matrices $M^{QP}_{(r)}$ define coefficients of eigenvectors of QP
(ii) (Completeness) Any eigenvector of QP comes from a coefficient eigenvector of $M^{QP}_{(r)}$ for some r.

Issue: Coefficient eigenvectors are orthogonal wrt
$W_r = [w(0), \ldots, w(K+1-r)]; \quad w_k = (k!)^2\binom{N-2r}{k}$

Problem: w_k are large numbers

Boolean analogue of prolate differential operator

$$\text{(BDO)} \qquad D(\alpha I - T^2)D + \beta T^2 \,.$$

T: diagonal; T^2: eigenvalues of Laplacian
$D = HTH$; $D^2 = L$.
HBDO$= H\mathrm{BDO}H$

Proposition

If $\beta = 2\sqrt{K(K+1)}$ then HBDO commutes with Q_K, almost commutes with P_K, and has tridiagonal, W-s.a. coefficient matrix M^{HBDO}

Eigenvectors of M^{HBDO} can be used as seeds for a weighted power method to compute coefficient eigenvectors of M_r^{QP}

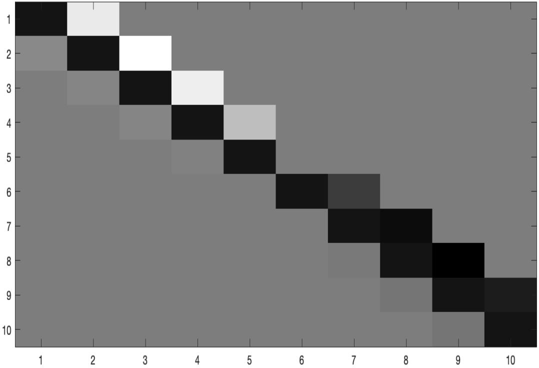


Figure: Matrix M^{HBDO}, $N = 9$, $K = 4$

PQ eigenvectors

PQ eigenvalues