Random Walks in Affine Weyl Groups and TASEPs on signed permutations

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Introduction

I will start to explain the setting for permutations and then turn to signed permutations.

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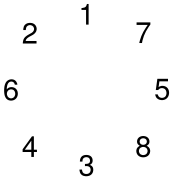

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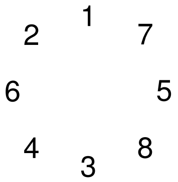

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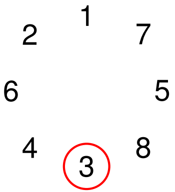

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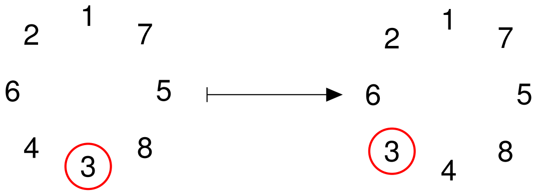

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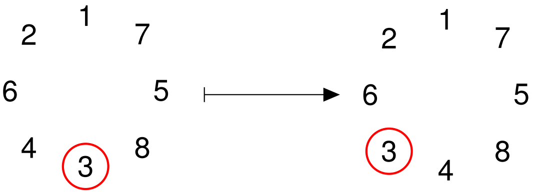

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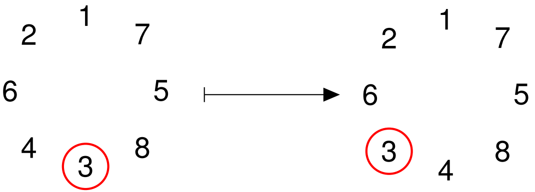

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This is an example of a TASEP (Totally Asymetric Simple Exclussion Process).

Example $n = 3$

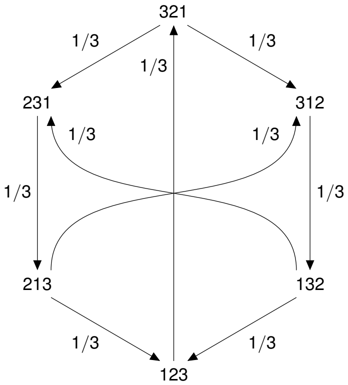

Figure: The cyclic-TASEP Markov chain for $n = 3$.

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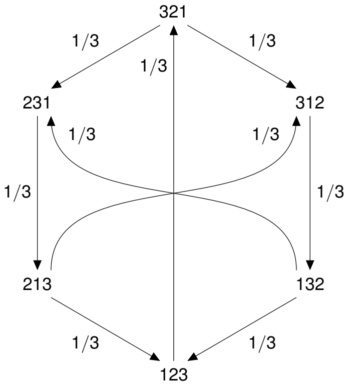

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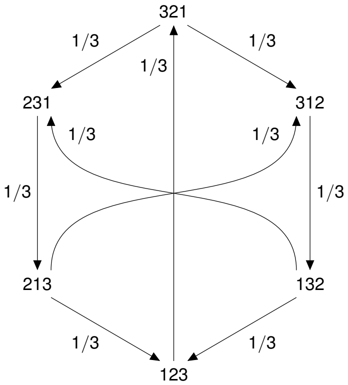

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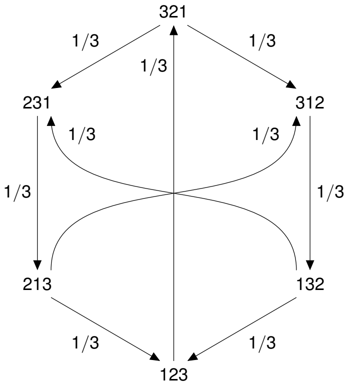

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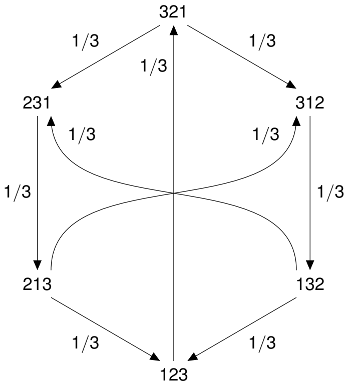

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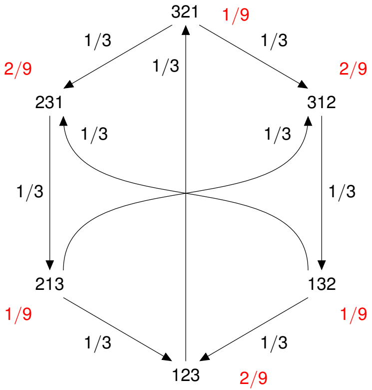

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For $n = 4$ we get

$p_{4321} = \frac{1}{96}$, $p_{4312} = \frac{3}{96}$
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Let $w_0 = n\ n-1 \dots 2\ 1$ be the reverse permutation.
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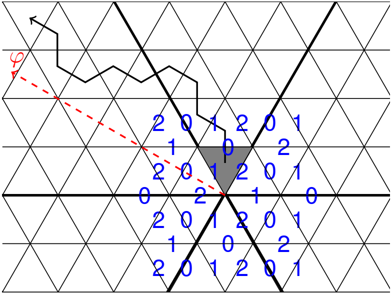
Theorem (Aas '12, Conjectured by Lam '11)

$$p_{id} = \frac{1}{2}, \frac{2}{9}, \frac{9}{96}, \frac{96}{2500}, \ldots \frac{\prod_i \binom{n-1}{i}}{\prod_i \binom{n}{i}}$$

Many results starting from this TASEP

This process has been studied from many perspectives by several different authors: Angel, Amir, Valko, Ferrari, Martin, Lam, Williams, Cantini, de Gier, Derrida, Ayyer, Corteel, Aas, Sjöstrand, De Sarkar, Evans, Arita, Prolhac, Mallick, Mandelshtam, Kim, Haglund, Mason, and probably others.

Connection to random walks



A reduced random walk in the alcoves of the $\tilde{A}_2$ arrangement.
The shown walk has reduced word $\cdots s_1 s_0 s_2 s_0 s_1 s_2 s_0 s_2 s_1 s_0$.
The thick lines divide V into Weyl chambers.

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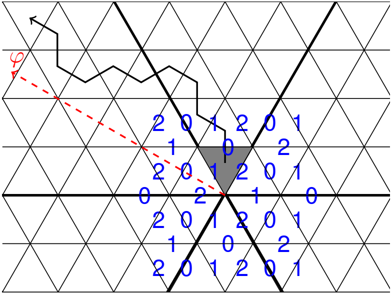

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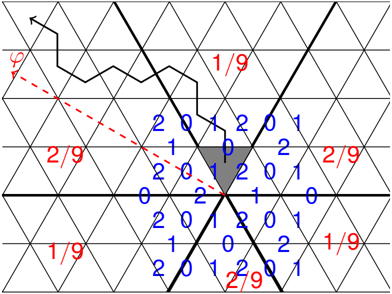

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The limiting direction φ is given by

$$\varphi = \frac{1}{Z} \sum_{w \in W : r_\theta w > w} p(w) w^{-1} \left(\theta^\vee\right),$$

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$$\frac{2\theta}{(\theta, \theta)}.$$

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2	2	0	7	3
3	4	2	0	6
4	6	4	2	0

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Theorem (Ayyer & L. '16)

For any $1 \leq i, j \leq n$,

$$c_{i,j} = \begin{cases} \frac{i-j}{n\binom{n}{2}}, & \text{if } i > j \\ 0, & \text{if } i = j \\ \frac{1}{n^2} + \frac{i(n-i)}{n^2(n-1)}, & \text{if } i = j-1 \\ \frac{1}{n^2}, & \text{if } i < j-1 \end{cases}$$

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Is needed to prove:

Theorem (Ayyer & L., Conjectured by Lam)

$\varphi = \sum_{1 \leq i < j \leq n} (e_i - e_j)$ *(the sum of all positive roots).*

Signed permutations

We now want to study the same problem for other Weyl groups.

Permutations with signs: $\begin{cases} B_n \\ C_n \end{cases}$

Permutations with an even number of signs: D_n

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I will focus on B_n today.

Alcoves and chambers

- Simple roots of B_n: $e_1, e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}$ and highest root $e_{n-1} + e_n$

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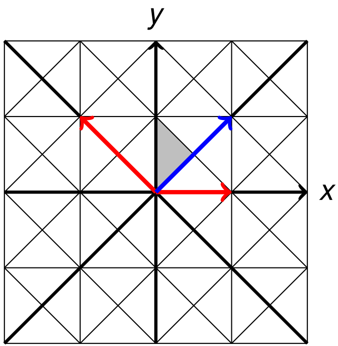

Lam's reduced random walk

Definition (Lam '15)

Begin at $X_0 = A^o$. Given $(X_0, X_1, \ldots, X_j)$, pick X_{j+1} at random among the alcoves adjacent to X_j, with the constraint that the hyperplane separating X_j and X_{j+1} has not been crossed.

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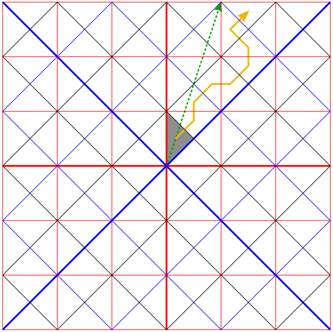
A reduced random walk in $\tilde{B}_2$ that stay in the fundamental chamber:



Kac labels as weights

Type	a_0	a_1	$\ldots a_i \ldots$	a_{n-1}	a_n
A	1	1	1	1	1
B	2	2	2	1	1
C	1	2	2	2	1
D	1	1	2	1	1

Type	$\check{a}_0$	$\check{a}_1$	$\ldots \check{a}_i \ldots$	$\check{a}_{n-1}$	$\check{a}_n$
$\check{B}$	1	2	2	1	1
$\check{C}$	1	1	1	1	1

Table: Kac-labels

B-multiTASEP

First site		Bulk		Last two sites	
Transition	Probability	Transition	Probability	Transition	Probability
$\bar{k} \to k$	$\frac{1}{n}$	$m\ell \to \ell m$	$\frac{1}{n}$	$j\,i \to \bar{i}\,\bar{j}$ $j\,i \to i\,j$ $j\,\bar{i} \to i\,\bar{j}$ $j\,\bar{i} \to \bar{i}\,j$ $i\,j \to \bar{j}\,\bar{i}$ $i\,\bar{j} \to \bar{j}\,i$ $\bar{i}\,j \to \bar{j}\,i$ $\bar{i}\,\bar{j} \to \bar{j}\,\bar{i}$	$\frac{1}{2n}$

Table: Transitions for the B-multiTASEP, where $\overline{n} \leq \ell < m \leq n$ and $1 \leq i < j, k \leq n$.

The Markov chain for B_2

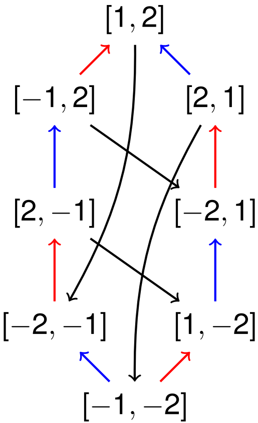


Figure: The Markov chain for B_2 as a multiTASEP on signed permutations.

multiTASEP of Type B

Theorem

The limiting direction of Lam's random walk on the alcoves of the affine Weyl group of type B_n, with probability rates weighted by the Kac-labels a_i is given by

$$\sum_{i=1}^{n}(2i-1)e_i. \quad \textit{again the sum of all positive roots}$$

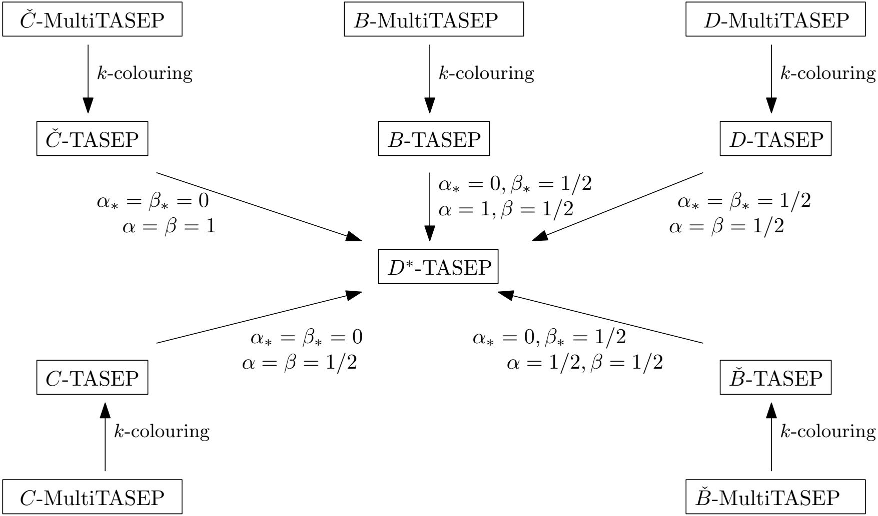
This is again proved using Lam's Theorem and studying correlation.

Correlations in Type B

$i \backslash j$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$
$\bar{4}$	0	$\frac{1}{32}$	$\mathbf{\frac{1}{64}}$	$\mathbf{\frac{1}{64}}$
$\bar{3}$	$\mathbf{\frac{1}{224}}$	0	$\frac{19}{448}$	$\mathbf{\frac{1}{64}}$
$\bar{2}$	$\mathbf{\frac{2}{224}}$	$\mathbf{\frac{1}{224}}$	0	$\frac{11}{224}$
$\bar{1}$	$\mathbf{\frac{3}{224}}$	$\mathbf{\frac{2}{224}}$	$\mathbf{\frac{1}{224}}$	0
1	$\mathbf{\frac{4}{224}}$	$\mathbf{\frac{3}{224}}$	$\frac{1}{32}$	0
2	$\mathbf{\frac{5}{224}}$	$\frac{3}{56}$	0	$\frac{1}{224}$
3	$\frac{13}{224}$	0	$\frac{1}{112}$	$\frac{3}{224}$
4	0	$\frac{3}{224}$	$\frac{5}{224}$	$\frac{3}{112}$

Table: The probabiity of i, j in the last two positions for B_4. The probability with j and $\bar{j}$ in the last position is the same, so only half the table is shown.

About the proofs



B-TASEP

First site		Bulk		Last two sites	
Transition	Probability	Transition	Probability	Transition	Probability
$\overline{1} \to 1$	$\frac{1}{n}$	$1\overline{1} \to \overline{1}1$ $10 \to 01$ $0\overline{1} \to \overline{1}0$	$\frac{1}{n}$	$11 \to \overline{11}$ $1\overline{1} \to \overline{1}1$ $01 \to \overline{1}0$ $0\overline{1} \to \overline{1}0$ $10 \to 0\overline{1}$ $10 \to 01$	$\frac{1}{2n}$

Table: Transitions for the *B*-TASEP.

D^*-TASEP

All our TASEPs have a further lumping to the D^*-TASEP with different parameters, where on each site we have exactly one particle from the set $\{*, 1, 0, \overline{1}\}$ subject to the following:

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First two sites		Bulk		Last two sites	
Transition	Probability	Transition	Probability	Transition	Probability
$*\overline{1} \to *1$	$\frac{\alpha}{n-1}$	$1\overline{1} \to \overline{1}1$	$\frac{1}{n-1}$	$1* \to \overline{1}*$	$\frac{\beta}{n-1}$
$*0 \to 01$	$\frac{\alpha_*}{n-1}$	$10 \to 01$		$0* \to \overline{1}0$	$\frac{\beta_*}{n-1}$
$0\overline{1} \to *0$	$\frac{1}{n-1}$	$0\overline{1} \to \overline{1}0$		$10 \to 0*$	$\frac{1}{n-1}$

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Two-row model

The stationary distribution of the D^*-TASEP may be described in terms of a Markov chain on certain two-row configurations (modification of a model by Duchi and Schaeffer).

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Example

Let $\widehat{\Omega}^*_{n,n_0}$ be the set of two-row configurations with n columns and n_0 0-columns.

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The stationary distribution of the D^*-TASEP may be described in terms of a Markov chain on certain two-row configurations (modification of a model by Duchi and Schaeffer). The two-row model lumps to the D^*-TASEP.

Example

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$$\widehat{\Omega}^*_{3,1} = \left\{ \begin{matrix} 0\,1\,* \\ 0\,\overline{1}\,* \end{matrix},\ \begin{matrix} 0\,\overline{1}\,* \\ 0\,1\,* \end{matrix},\ \begin{matrix} *\,0\,* \\ *\,0\,* \end{matrix},\ \begin{matrix} *\,1\,0 \\ *\,\overline{1}\,0 \end{matrix},\ \begin{matrix} *\,\overline{1}\,0 \\ *\,1\,0 \end{matrix} \right\}$$

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and

$$\widehat{\Omega}^*_{4,0} = \left\{ \begin{matrix} *\,\overline{1}\,\overline{1}\,* \\ *\,1\,1\,* \end{matrix}, \begin{matrix} *\,\overline{1}\,1\,* \\ *\,1\,\overline{1}\,* \end{matrix}, \begin{matrix} *\,1\,\overline{1}\,* \\ *\,1\,\overline{1}\,* \end{matrix}, \begin{matrix} *\,1\,\overline{1}\,* \\ *\,\overline{1}\,1\,* \end{matrix}, \begin{matrix} *\,1\,1\,* \\ *\,\overline{1}\,\overline{1}\,* \end{matrix} \right\}.$$

The transitions are tedious to describe.

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- Two-row configurations without 0's are in bijection with bicolored Motzkin paths and Dyck paths, so computing $\langle i, j \rangle$ in the D^*-TASEP reduces to counting paths with weights.
- Although we don't have enough information left to compute the $\langle i, j \rangle$ in the original multiTASEPs, it aloows us e.g. to compute the sum

$$\sum_{j=i+1}^{n} \langle j, i \rangle - \langle j, \bar{i} \rangle + \langle i, j \rangle - \langle \bar{i}, j \rangle$$

  Enough to determine the limiting direction for $\tilde{B}_n$.

$\check{C}$-multiTASEP

I mention one last result

Theorem

The limiting direction of Lam's random walk on the alcoves of the affine Weyl group of type C_n, weighted by the dual Kac-labels $\check{a}_i$ is given by

$$\sum_{i=1}^{n}(2i+1)e_i. \quad \text{(the sum of positive roots is however } \sum_i (2i)e_i\text{)}$$

Thanks for your attention!