Descents on quasi-Stirling permutations

Sergi Elizalde

Dartmouth College

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Descents

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$\mathrm{des}(36{\cdot}5{\cdot}22{\cdot}13{\cdot}1{\cdot}) = 5$

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Example

$A_1(t) = t$ $\quad$ $1\cdot$

$A_2(t) = t + t^2$ $\quad$ $12\cdot, 2\cdot1\cdot$

$A_3(t) = t + 4t^2 + t^3$ $\quad$ $123\cdot, 13\cdot2\cdot, 2\cdot13\cdot, 23\cdot1\cdot, 3\cdot12\cdot, 3\cdot2\cdot1\cdot$

$A_4(t) = t + 11t^2 + 11t^3 + t^4$ $\quad$ $\ldots$

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These polynomials appear in work of Euler from 1755.

Eulerian polynomials

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3\,(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4\,(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5\,(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1.2.3.4.5.6\,(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1.2.3.4.5.6.7\,(p-1)^7}$$

&c.

L. Euler, 1755.

Eulerian Polynomials

$$\frac{A_n(p)/p}{n!(p-1)^n} \quad (1 \le n \le 7)$$

Eulerian polynomials

Euler was considering the series

$$\sum_{m\geq 0} m t^m = \frac{t}{(1-t)^2}$$

$$\sum_{m\geq 0} m^2 t^m = \frac{t+t^2}{(1-t)^3}$$

$$\sum_{m\geq 0} m^3 t^m = \frac{t+4t^2+t^3}{(1-t)^4}$$

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In general,

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

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In general,

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This can be proved by induction on n, differentiating both sides.

Generating function for Eulerian polynomials

Let

$$A(t,z)=\sum_{n\geq 0} A_n(t)\frac{z^n}{n!}$$

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We have

$$A(t,z)=\frac{1-t}{1-te^{(1-t)z}}.$$

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Example: $S(3, 2) = 3$. $\{1, 2\} \cup \{3\}$, $\{1, 3\} \cup \{2\}$, $\{2, 3\} \cup \{1\}$.

In 1978, Gessel and Stanley considered the series

$$\sum_{m \geq 0} S(m+1, m)\, t^m = \frac{t}{(1-t)^3}$$

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$$\sum_{m \geq 0} S(m+3, m)\, t^m = \frac{t + 8t^2 + 6t^3}{(1-t)^7}$$

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What are the polynomials in the numerator? Positive coefficients?

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We have $|\mathcal{Q}_n| = (2n-1)\cdot(2n-3)\cdot\cdots\cdot 3\cdot 1$, since every permutation in $\mathcal{Q}_n$ can be obtained by inserting nn into one of the $2n-1$ spaces of a permutation in $\mathcal{Q}_{n-1}$.

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Theorem (Gessel–Stanley '78)

$$\sum_{m\geq 0} S(m+n, m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}}.$$

Literature on Stirling permutations

There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- Bóna '08: $Q_n(t)$ also gives the enumeration of $\mathcal{Q}_n$ by the number of plateaus, that is, positions i such that $\pi_i = \pi_{i+1}$.

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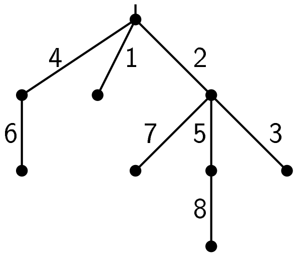
- Janson '08: The joint distribution of ascents, descents and plateaus on $\mathcal{Q}_n$ is asymptotically normal.
- The coefficients of $Q_n(t)$ are sometimes called second-order Eulerian numbers.

Stirling permutations and trees

$\mathcal{I}_n$ = set of increasing edge-labeled plane rooted trees with n edges.



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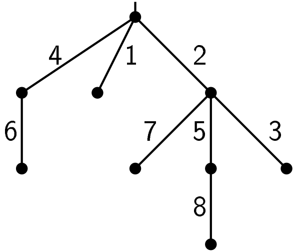

$\xrightarrow{\varphi}$ 4664112775885332

Theorem (Koganov '96, Janson '08)

There is a bijection $\varphi : \mathcal{I}_n \longrightarrow \mathcal{Q}_n$ obtained by traversing the edges of the tree along a depth-first walk from left to right, and recording their labels.

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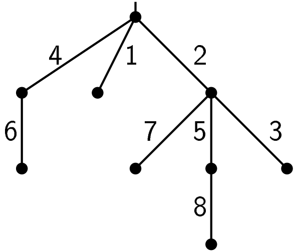

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If we remove the increasing condition on the trees, what is the image of φ?

Removing the increasing condition

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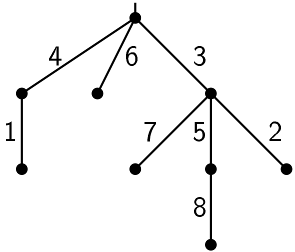

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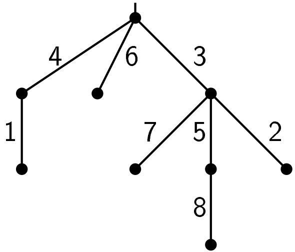

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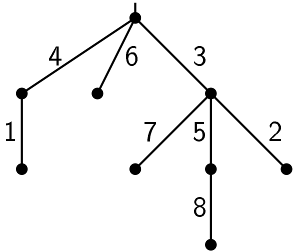

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Theorem (Archer–Gregory–Pennington–Slayden '19)

φ is a bijection between $\mathcal{T}_n$ and $\overline{\mathcal{Q}}_n$ (to be defined in the next slide).

Quasi-Stirling permutations

Definition (Archer–Gregory–Pennington–Slayden '19)

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Example

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The number of unlabeled plane rooted trees with n edges is the Catalan number C_n. It follows from the bijection that

$$|\overline{\mathcal{Q}}_n| = n!\,C_n = \frac{(2n)!}{(n+1)!}.$$

Descents on quasi-Stirling permutations

Conjecture (Archer–Gregory–Pennington–Slayden '19)

The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\mathrm{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

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Example

Set of $\pi \in \overline{\mathcal{Q}}_3$ with $\mathrm{des}(\pi) = 1$: $\{112233\}$ 1

with $\mathrm{des}(\pi) = 2$: 13

$\{112332, 113223, 113322, 122133, 122331, 133122, 211233, 221133, 223113, 223311, 233112, 311223, 331122\}$

with $\mathrm{des}(\pi) = 3$: 16

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Conjecture (Archer–Gregory–Pennington–Slayden '19)

The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\operatorname{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

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To prove this conjecture, we look at how descents are transformed by the bijection φ.

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Lemma

If $T \in \mathcal{T}_n$ *and* $\pi = \varphi(T) \in \overline{\mathcal{Q}}_n$, *then*

$$\operatorname{des}(\pi) = \operatorname{cdes}(T),$$

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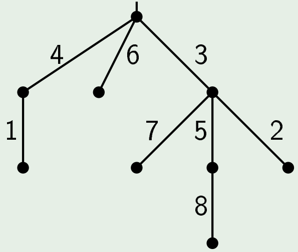

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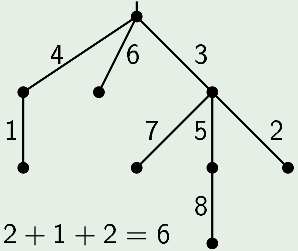

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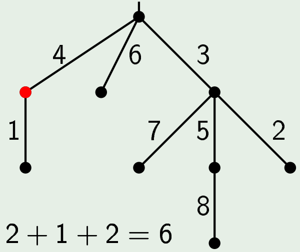

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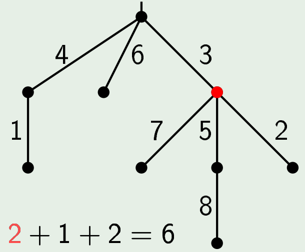

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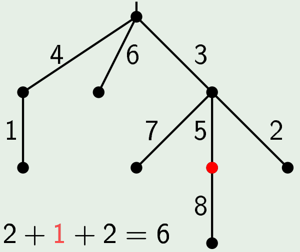

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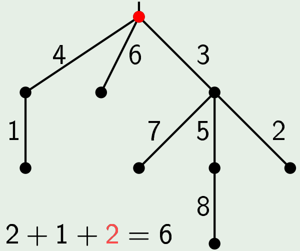

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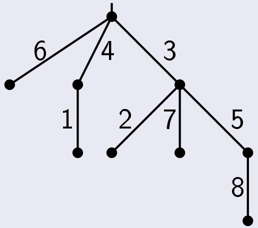

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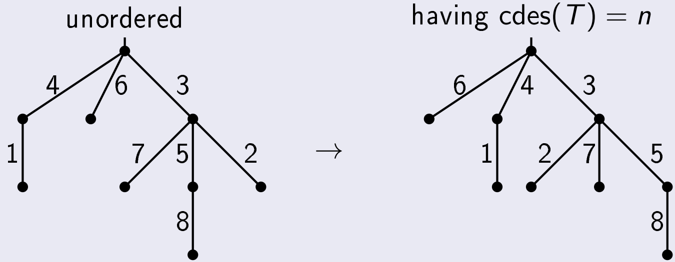

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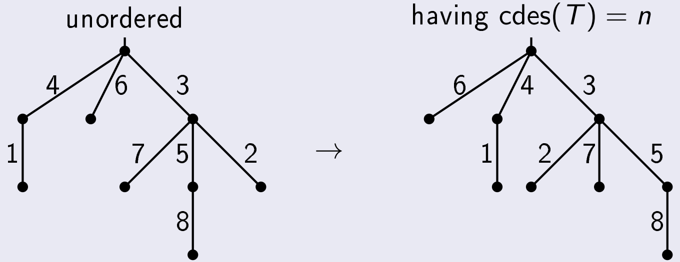

By Cayley's formula, there are $(n+1)^{n-1}$ such trees. □

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Recall the Eulerian polynomials $A_n(t) = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{des}(\pi)}$ and their EGF

$$A(t,z) = \sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{1-te^{(1-t)z}}.$$

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The EGF $\overline{Q}(t,z)$ *for quasi-Stirling permutations by the number of descents satisfies the implicit equation*

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Its coefficients satisfy

$$\overline{Q}_n(t) = \frac{n!}{n+1}\,[z^n]A(t,z)^{n+1}.$$

Here $[z^n]F(z)$ denotes the coefficient of z^n in $F(z)$.

Proof ideas

By the bijection φ,

$$\overline{Q}(t,z) = \sum_{n\geq 0} \sum_{\pi \in \overline{\mathcal{Q}}_n} t^{\mathrm{des}(\pi)} \frac{z^n}{n!} = \sum_{n\geq 0} \sum_{T \in \mathcal{T}_n} t^{\mathrm{cdes}(T)} \frac{z^n}{n!}.$$

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Decompose trees in $\mathcal{T}_n$ as

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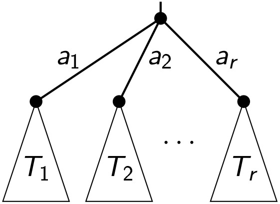

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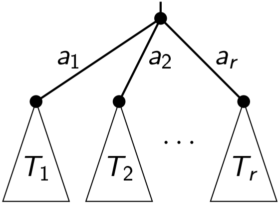
$T =$ 

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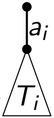
$\mathrm{cdes}(T) = \sum_{i=1}^{r} (\mathrm{cdes}($ 

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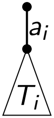
The EGF for each piece 

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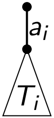
The EGF for each piece 

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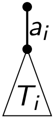
The EGF for each piece 

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Finally, extracting its coefficients using Lagrange inversion gives

$$\overline{Q}_n(t) = \frac{n!}{n+1}\,[z^n]A(t,z)^{n+1}.$$

Consequences

Recall the formulas:

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

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Properties of quasi-Stirling polynomials

Recall: i is a descent of π if $\pi_i > \pi_{i+1}$ or $i = r$,
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- *The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.*
- *The distribution of the number of descents on $\overline{\mathcal{Q}}_n$ is asymptotically normal.*

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Note: $\mathcal{Q}_n^1 = \overline{\mathcal{Q}}_n^1 = \mathcal{S}_n, \quad \mathcal{Q}_n^2 = \mathcal{Q}_n, \quad \overline{\mathcal{Q}}_n^2 = \overline{\mathcal{Q}}_n.$

Enumeration of k-Stirling and k-quasi-Stirling permutations

Counting k-Stirling permutations is easy, since every permutation in $\mathcal{Q}_n^k$ can be obtained by inserting the string $n^k = nn\dots n$ into one of the $(n-1)k+1$ spaces of a permutation in $\mathcal{Q}_{n-1}^k$, so

$$|\mathcal{Q}_n^k| = (k+1)(2k+1)\cdot\dots\cdot((n-1)k+1).$$

Enumeration of k-Stirling and k-quasi-Stirling permutations

Counting k-Stirling permutations is easy, since every permutation in $\mathcal{Q}_n^k$ can be obtained by inserting the string $n^k = nn\dots n$ into one of the $(n-1)k+1$ spaces of a permutation in $\mathcal{Q}_{n-1}^k$, so

$$|\mathcal{Q}_n^k| = (k+1)(2k+1)\cdot\dots\cdot((n-1)k+1).$$

Theorem

For $n \geq 1$ and $k \geq 1$,

$$|\overline{\mathcal{Q}}_n^k| = \frac{(kn)!}{((k-1)n+1)!} = n!\, C_{n,k},$$

where

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

is the nth k-Catalan number.

k-quasi-Stirling permutations and trees

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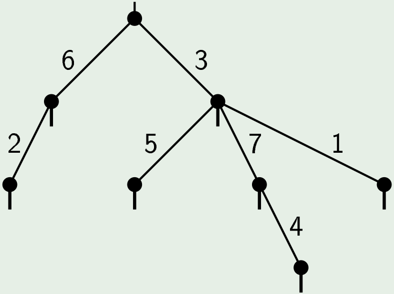
Example

A bijection between *compartmented trees* and 3-quasi-Stirling permutations:



$\xrightarrow{\phi}$ 62226635553774447113

Ascents, descents and plateaus on k-quasi-Stirling permutations

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Define the multivariate k-quasi-Stirling polynomials

$$\overline{P}_n^{(k)}(q,t,u) = \sum_{\pi \in \overline{\mathcal{Q}}_n^k} q^{\mathrm{asc}(\pi)} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

and their EGF

$$\overline{P}^{(k)}(q,t,u;z) = \sum_{n \geq 0} \overline{P}_n^{(k)}(q,t,u) \frac{z^n}{n!}.$$

Ascents, descents and plateaus on k-quasi-Stirling permutations

This is the most general version of our main result:

Theorem

$\overline{P}^{(k)}(q,t,u;z)$ satisfies the implicit equation

$$\overline{P}^{(k)}(q,t,u;z) = 1 - q + \frac{q(q-t)}{q - te^{(q-t)z(\overline{P}^{(k)}(q,t,u;z)-1+u)^{k-1}}}.$$

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$$\overline{P}_n^{(k)}(q,t,u) = \frac{n!}{(k-1)n+1}\,[z^n]\left(u-q+\frac{q(q-t)}{q-te^{(q-t)z}}\right)^{(k-1)n+1}.$$

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The proof follows ascents, descents and plateaus through the bijection ϕ, and it uses a decomposition of compartmented trees.

Further research

Quasi-Stirling permutations of $\{1,1,2,2,\dots,n,n\}$ avoid 1212 and 2121, so we can think of them as labeled *non-crossing* matchings.

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Thank you