Metric Invariants of Spherical Harmonics

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Spherical Harmonics

1. Spherical harmonics of degree k are smooth solutions of the Euler

$$xu_x + yu_y + zu_z - ku = 0,$$

and the Laplace

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the correspondig equations and their prolongations.

3. Lie group $\mathbf{SO}(3)$ is the obvious symmetry group of these equations and all $\mathcal{E}^{(i)}$ are affine algebraic manifolds equipped with the algebraic $\mathbf{SO}(3)$ $-$action.

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2. The Lie algebra $\mathfrak{sl}(2) \subset \mathbb{A}_3$, is generated by the following operators

$$X_+ = \frac{r^2}{2}, H = \delta + \frac{3}{2}, X_- = \frac{\Delta}{2},$$

where

$$r^2 = x^2 + y^2 + z^2,\ \delta = x\partial_x + y\partial_y + z\partial_z,\ \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2,$$

and operators (X_+, H, X_-) form the Weyl basis in $\mathfrak{sl}(2)$:
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$[H, X_+] = 2X_+, [H, X_-] = -2X_-, [X_-, X_+] = H.$
3. The Lie algebra $\mathfrak{so}(3) \subset \mathbb{A}_3$ generated by the angular momentum operators

$$L_z = x\partial_y - y\partial_x, L_y = x\partial_z - z\partial_x, L_x = y\partial_z - z\partial_y.$$

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2. *Casimir operator in Lie algebra* $so(3)$ is the orbital angular momentum operator

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3. The restriction of operator M on the unit sphere is the spherical Laplace operator.

Harmonic polynomials

1. The following sequence

$$0 \longrightarrow \mathbb{H}_k \longrightarrow \mathbb{P}_k \xrightarrow{\Delta} \mathbb{P}_{k-2} \longrightarrow 0$$

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2. *Splitting* $\mathbb{P}_k$: for any homogeneous polynomial $p_k \in \mathbb{P}_k$ there are (and unique) spheric harmonics $h_{k-2i} \in \mathbb{H}_{k-2i}$, $0 \leq i \leq [\frac{k}{2}]$, such that

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4. The restriction of spheric harmonics on the unit sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ are eigenfunctions of the spherical laplacian Δ_S with eigenvalues $-k(k+1)$ and any continuous function on $\mathbf{S}^2$ could be approximated (with any accuracy) by linear combination of spherical harmonics.

Harmonic projections

1. *Harmonic projections* $\eta_{k,2i} : \mathbb{P}_k \to \mathbb{H}_{k-2i}$ are the following

$$\eta_{k,2i} = r^{-2i} Q_{k,2i}\left(M\right),$$

where

$$Q_{k,2i}\left(\lambda\right) = \prod_{j\neq i}^{\left[\frac{k}{2}\right]} \frac{\lambda - \lambda_j}{\lambda_i - \lambda_j}, \quad \lambda_i = -\left(k-2i\right)\left(k-2i+1\right).$$

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4. Here
$$\eta_{k+l,0} = \prod_{j=1}^{\left[\frac{k+l}{2}\right]} \frac{M + (k+l-2j)(k+l-2j+1)}{2j(2j-2k-2l-1)}.$$

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4. Example.
$$x * x = xx - \frac{r^2}{3}, x * y = xy.$$

Algebraic invariants

1. The space $\mathbb{H}_k$ of spherical harmonics is a vector space of dimension $2k+1$. The Lie group $\mathbf{SO}(3)$ acts in algebraic way on $\mathbb{H}_k$, and in $\mathbb{H}_k$, all irreducible representations of the group $\mathbf{SO}(3)$ are realized .

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3. Due to Rosenlicht theorem rational invariants of this action (i.e. rational invariants of spherical harmonics) form a field of trancedence degree equals to the codimension of regular orbit.
4. Regular orbit has codimension $(2k-2)$,when $k \geq 2$, and codimention 1, when $k=1$. Therefore, in order to define a regular orbit, we need $2k-2$ algebraicly independent rational invariants, for $k>2$, and only one invariant, for $k=1$.

Differential invariants

1. Equations $\mathcal{E}^{(i)}$ are affine manifolds of dimension $2i+4$, if $2 \leq i < k$. The regular $\mathbf{SO}(3)$ $-$orbits correspond to smooth points of the quotient $\mathcal{E}^{(i)}/\mathbf{SO}(3)$) Thus, due to Hilbert theorem, the quotients are affine manifolds of dimension $2i+1$. Rational differential invariants of order $\leq i$ are rational functions on $\mathcal{E}^{(i)}/\mathbf{SO}(3)$ and therefore the trancedence degree of field $\mathcal{F}_i^d$ equals to $2i+1$.

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2. As we have seen, the trancedence degree of field $\mathcal{F}_k^a$ equals $2(k-1)$.
3. Take a regular harmonic $h \in \mathbb{H}_k$. Then it is easy to check that the $\mathbf{SO}(3)$ $-$orbit of the 2-jet $j_2(h)$ into $\mathcal{E}^{(2)}$ is a 6-dimensional submanifold into 8-dimensional manifold $\mathcal{E}^{(2)}$ and therefore *we need 2 differential invariants of order 2 to describe the orbit* (compare with $2(k-1)$ algebraic invariants).

Iinvariant coframe

1. Total differentials of the obvious invariants $J_{-1} = \frac{r^2}{2}$ and $J_0 = u$ give us two $\mathbf{SO}(3)$-invariant horizontal $1-$forms:

$$\begin{aligned} \omega_1 &= x dx + y dy + z dz, \\ \omega_2 &= u_x dx + u_y dy + u_z dz. \end{aligned}$$

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3. Then coframe $(\omega_1, \omega_2, \omega_3)$ is $\mathbf{SO}(3)$-invariant.

Iinvariant frame

1.

$$
\begin{aligned}
D_1 &= x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz},\\
D_2 &= u_x\frac{d}{dx} + u_y\frac{d}{dy} + u_z\frac{d}{dz},\\
D_3 &= (yu_z - zu_y)\frac{d}{dx} + (zu_x - xu_z)\frac{d}{dy} + (xu_y - yu_x)\frac{d}{dz}.
\end{aligned}
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First invariants

$$
\begin{aligned}
J_{-1} &= \frac{r^2}{2}, J_0 = u, \\
J_1 &= D_2\left(J_0\right) = u_x^2 + u_y^2 + u_z^2, \\
J_{21} &= \frac{D_2\left(J_1\right)}{2} = u_x^2 u_{xx} + u_y^2 u_{yy} + u_z^2 u_{zz} + \\
& 2\left(u_x u_y u_{xy} + u_x u_z u_{xz} + u_y u_z u_{yz}\right).
\end{aligned}
$$

Invariant symmteric forms and operators

1. Symmetric differenatial i-forms

$$\Theta_i = \sum_{i_1+i_2+i_3=i} u_{i_1,i_2,i_3} \frac{dx^{i_1}}{i_1!} \cdot \frac{dy^{i_2}}{i_2!} \cdot \frac{dz^{i_3}}{i_3!}$$

are invariants with respect to Lie group of affine transformations in $\mathbb{R}^3$.

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are invariants with respect to Lie group of affine transformations in $\mathbb{R}^3$.

2. Differential operators

$$\widehat{\Theta}_i = \sum_{i_1+i_2+i_3=i} \frac{u_{i_1,i_2,i_3}}{i_1!i_2!i_3!} \frac{d^k}{dx^{i_1}dy^{i_2}dz^{i_3}}$$

are $\mathbf{SO}(3)$-invariant.

Invariants

Let

$$\begin{aligned} dx &= t_{11}\omega_1 + t_{12}\omega_2 + t_{13}\omega_3, \\ dy &= t_{21}\omega_1 + t_{22}\omega_2 + t_{23}\omega_3, \\ dz &= t_{31}\omega_1 + t_{32}\omega_2 + t_{33}\omega_3, \end{aligned}$$

where t_{ij} are rational functions on $J^1\left(\mathbb{R}^3\right)$, and let

$$\Theta_i = \sum_{i_1+i_2+i_3=i} T_{i_1,i_2,i_3} \frac{\omega_1^{i_1}}{i_1!} \cdot \frac{\omega_2^{i_2}}{i_2!} \cdot \frac{\omega_3^{i_3}}{i_3!}.$$

Theorem

Functions T_{i_1,i_2,i_3} are rational differential **SO** (3)*-invariants of order $i = i_1 + i_2 + i_3$ and any rational differential* **SO** (3)*-invariants of order i is a rational function of them.*

Example

Remark that invariants

$$G_i = \widehat{\Theta}_i\,(u) = \sum_{i_1+i_2+i_3=i} \frac{u^2_{i_1,i_2,i_3}}{i_1!i_2!i_3!}$$

are squares of lengths of symmetric forms Θ_i.
Thus,

$$\begin{aligned}
\widehat{\Theta}_1 &= u_x \frac{d}{dx} + u_y \frac{d}{dy} + u_z \frac{d}{dz}, \\
\widehat{\Theta}_2 &= \frac{1}{2}\left(u_{xx}\frac{d^2}{dx^2} + u_{yy}\frac{d^2}{dy^2} + u_{zz}\frac{d^2}{dz^2}\right) + u_{xy}\frac{d^2}{dxdy} + u_{xz}\frac{d^2}{dxdz} + u_{yz}\frac{d^2}{dydz}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\Theta}_1\,(u) &= u_x^2 + u_y^2 + u_z^2, \\
\widehat{\Theta}_1\,(u) &= J_{22} = \frac{u_{xx}^2 + u_{yy}^2 + u_{zz}^2}{2} + u_{xy}^2 + u_{xz}^2 + u_{yz}^2.
\end{aligned}$$

Theorem

The field of rational differential $\mathbf{SO}(3)$*-invariants of spherical harmonics is generated by invariants* $\left(J_{-1} = \frac{r^2}{2}, J_0 = u, J_{22}\right)$ *and derivation* $\nabla = \widehat{\Theta}_1$.

Monoid of invariants

1. **SO** (3) −Invariants $\Longleftrightarrow$ $SO(3)$ −invariant differential operators:

$$\phi \in C^{\infty}\left(\mathbf{J}^{k}\left(\mathbb{R}^{3}\right)\right) \Longleftrightarrow \Delta_{\phi} : C^{\infty}\left(\mathbb{R}^{3}\right) \to C^{\infty}\left(\mathbb{R}^{3}\right),$$

$$\Delta_{\phi}(f) = j_{k}(f)^{*}(\phi).$$

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2. Monoid structure on **SO** (3) −invariants is defined by the composition of invariant operators, and $\mathrm{id} = u$.
3. Thus, the field $\mathcal{F}_{k}^{d}$ is the monoid.

Weights

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2. We say that a polynomial differential invariant I has weight $w(I)$ if
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3. In other words, if h is a homogeneous polynomial of degree k then $I(h)$ has degree $w(I)$.

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1. Algebraic invariants on $\mathbb{H}_k$ are differential invariants of order k.
2. Let I be a polynomial differential invariant of weight w, and $h \in \mathbb{H}_k$.
3. Then $I(h) \in \mathbb{P}_w$, $(\eta_{w,2l} \circ \Delta_I)(h) \in \mathbb{H}_{w-2l}$ and its length $(\Delta_{G_{w-2l}} \circ \eta_{w,2l} \circ \Delta_I)(h)$ is a scalar, i.e invariant $G_{w-2l} \circ \eta_{w,2l} \circ I$ is an algebraic invariant.