Symmetries and Noether's conservation laws of semi-discrete equations

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Review and motivations

A brief introduction to symmetries of DEs

Symmetries of DDEs

Noether's theorems for DDEs

Summary

Symmetries of DDEs: a brief review

- Finite difference equations: S. Maeda (1980s), Vladimir Dorodnitsyn (1990s–), Peter Hydon & Elizabeth Mansfield (2000s–), ...

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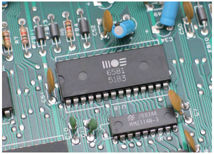
Challenge for DDEs: the **noncommutativity** (that we will see shortly)

- [Levi–Winternitz–Yamilov, 2010]: Lie point symmetries of differential-difference equations, *Journal of Physics A: Mathematical and Theoretical* **43**, 292002.
- [P, 2017]: Symmetries, Conservation Laws, and Noether's Theorem for Differential-Difference Equations, *Studies in Applied Mathematics* **139**, 457–502.
- [P–Hydon, 2021]: Transformations, symmetries and Noether theorems for differential-difference equations, *preprint*.

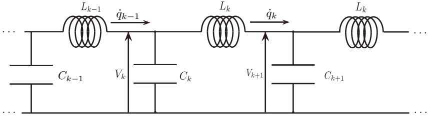
Motivations

Why is the study of semi-discrete equations important?

- Semi-discretization of PDEs and semi-continuum of PΔEs
- They naturally arise as models of mechanical or physical systems, e.g., Toda lattice, Volterra equations, interconnected mechanical systems



LSI Circuit

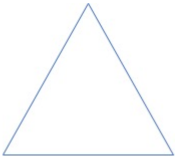


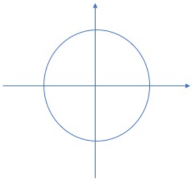
Transmission Line

What is a symmetry (or symmetry group)?

Planar or 3D objects: A local diffeomorphism of transformation which preserves the structure and the shape.

- Rotation of an equilateral triangle by $2k\pi/3$ for any integer $k \in \mathbb{Z}$: a discrete symmetry





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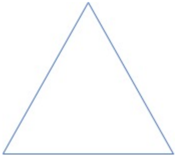

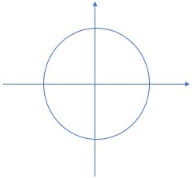

- Consider the unit circle $x^2 + y^2 = 1$. The transformation Γ_ε is

$$\Gamma_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \Gamma_0 = \mathrm{id}\,.$$

  The infinitesimal generator with respect to Γ_ε is

$$\mathbf{v} = \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \widetilde{x} \right) \partial_x + \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \widetilde{y} \right) \partial_y = -y\partial_x + x\partial_y.$$

Symmetries of DEs

For the unit circle $x^2 + y^2 = 1$, we notice that after transformation Γ_ε we have

$$\widetilde{x}^2 + \widetilde{y}^2 = (x^2 + y^2) = 1.$$

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Example. Consider the Riccati equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y+1}{x} + \frac{y^2}{x^3}$$

and the transformation

$$\Gamma_\varepsilon : (x, y) \mapsto \left(\widetilde{x} = \frac{x}{1-\varepsilon x}, \widetilde{y} = \frac{y}{1-\varepsilon x}\right).$$

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$$\Gamma_\varepsilon : (x, y) \mapsto \left(\widetilde{x} = \frac{x}{1 - \varepsilon x}, \widetilde{y} = \frac{y}{1 - \varepsilon x}\right).$$

- Direct calculation shows that

$$\widetilde{y}' = \frac{\widetilde{y}+1}{\widetilde{x}} + \frac{\widetilde{y}^2}{\widetilde{x}^3}$$

Prolongation of transformations and the LSC

For $\Gamma_\varepsilon : (x, y) \mapsto (\widetilde{x}, \widetilde{y})$, the chain rule gives

$$\widetilde{y}' = \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\widetilde{x}} = \frac{D_x \widetilde{y}}{D_x \widetilde{x}}, \quad \dots$$

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To determine symmetries of $y' - w(x, y) = 0$ using the **linearized symmetry condition (LSC)**:

1. Taylor expansion of $\widetilde{y}' - w(\widetilde{x}, \widetilde{y}) = 0$:

   $$y' - w(x, y) + \varepsilon(\phi_x + (\phi_y - \xi_x)y' - \xi_y y'^2 - \xi w_x - \phi w_y) + O(\varepsilon^2) = 0,$$

   where

   $$\widetilde{x} = x + \varepsilon\xi(x, y) + O(\varepsilon^2), \quad \widetilde{y} = y + \varepsilon\phi(x, y) + O(\varepsilon^2).$$

2. Using the infinitesimal generator $\mathbf{v} = \xi\partial_x + \phi\partial_y$:

   $$\mathbf{prv}(y' - w(x, y)) = 0 \text{ whenever } y' = w(x, y),$$

   where

   $$\mathbf{prv} = \mathbf{v} + (D_x(\phi - \xi y') + \xi y'')\,\partial_{y'} + \cdots .$$

In both cases: prolongation of transformations is essential.

- For a transformation $\Gamma_\varepsilon : (x, y) \mapsto (\widetilde{x}(\varepsilon, x, y), \widetilde{y}(\varepsilon, x, y))$ s.t. $\Gamma_0 = \mathrm{id}$, prolong the transform to derivatives

$$\widetilde{y}' = \frac{\mathrm{d}\widetilde{y}}{\mathrm{d}\widetilde{x}} = \frac{D_x \widetilde{y}}{D_x \widetilde{x}}, \quad \widetilde{y}'' = \frac{D_x \widetilde{y}'}{D_x \widetilde{x}}, \quad \ldots$$

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- The infinitesimal generator of Γ_ε is $\mathbf{v} = \xi\partial_x + \phi\partial_y$ where

$$\xi = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\widetilde{x}, \quad \phi = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\widetilde{y}.$$

  Its prolongation is naturally $\mathbf{pr v} = \mathbf{v} + \phi^1\partial_{y'} + \phi^2\partial_{y''} + \cdots$ where

$$\phi^1 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\widetilde{y}', \quad \phi^2 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\widetilde{y}'', \quad \ldots$$

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- The general prolongation formula is equivalent to an evolutionary representative

$$\mathbf{pr v} = \xi D_x + Q\partial_y + (D_x Q)\partial_{y'} + \cdots, \quad Q(x, y, y') = \phi - \xi y'.$$

Symmetries of DDEs

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- Shorthand notations:

$$u = u(x,n), \;\; u_j = u(x,n+j), \;\; u' = D_x u(x,n), \;\; u'_j = D_x u(x,n+j), \;\; \ldots$$

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$$u = u(x,n),\ \ u_j = u(x,n{+}j),\ \ u' = D_x u(x,n),\ \ u_j' = D_x u(x,n{+}j),\ \ \dots$$

- **Noncommutativity** [P, 2017]: how to prolong a transformation

$$\Gamma_\varepsilon : (x,n,u) \mapsto (\widetilde{x}(\varepsilon,x,n,u), n, \widetilde{u}(\varepsilon,x,n,u));$$

namely, how to calculate, for instance

$$\widetilde{u}_1 = \widetilde{u}(\varepsilon,x,n+1,u) \text{ or } \widetilde{u}(\varepsilon,x,n+1,u_1)?$$
$$\widetilde{u}_1' = \ ? \text{ (shift first or differentiate first?)}$$

Example. Consider the following local transformations

$$\widetilde{x} = x + \varepsilon u, \quad \widetilde{u} = u.$$

- Then we have ($S : n \mapsto n+1$: forward shift)

$$D_{\widetilde{x}}\widetilde{u} = \frac{D_x\widetilde{u}}{D_x\widetilde{x}} = \frac{u_x}{1+\varepsilon u_x},$$
$$S(D_{\widetilde{x}}\widetilde{u}) = \frac{Su_x}{1+\varepsilon Su_x},$$

and

$$S\widetilde{u} = Su = u(x, n+1),$$
$$D_{\widetilde{x}}(S\widetilde{u}) = \frac{D_x(S\widetilde{u})}{D_x\widetilde{x}} = \frac{Su_x}{1+\varepsilon u_x}.$$

- Apparently $S(D_{\widetilde{x}}\widetilde{u}) \neq D_{\widetilde{x}}(S\widetilde{u})$; which one is $\widetilde{u}_1'$?

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Example continued. Consider the following local transformations

$$\widetilde{x} = x + \varepsilon u, \quad \widetilde{u} = u.$$

- $(x, n, u) \Leftrightarrow (S, D = D_x)$ and $(\widetilde{x}, \widetilde{n}, \widetilde{u}) \Leftrightarrow (\widetilde{S}, \widetilde{D} = D_{\widetilde{x}})$
  Certainly $\widetilde{D}\widetilde{S} = \widetilde{S}\widetilde{D}$
- The calculation of $\widetilde{u}_1'$ for $u = u(x, n)$:

$$\begin{aligned}
\widetilde{u}_1' = \widetilde{u}'(\widetilde{x}, \widetilde{n}+1) = \widetilde{S}(\widetilde{D}\widetilde{u}(\widetilde{x}, \widetilde{n})) &= \widetilde{S}(\widetilde{D}u(x, n)) \\
&= \widetilde{S}(\widetilde{D}u(\widetilde{x} - \varepsilon\widetilde{u}, \widetilde{n})) \\
&= \widetilde{S}\left(u'(x, n) \cdot (1 - \varepsilon\widetilde{u}'(\widetilde{x}, \widetilde{n}))\right) \\
&= u'(\widetilde{x} - \varepsilon\widetilde{u}_1, \widetilde{n}+1) \cdot (1 - \varepsilon\widetilde{u}'(\widetilde{x}, \widetilde{n}+1))
\end{aligned}$$

$$\therefore \quad \widetilde{u}_1' = \frac{u'(\widetilde{x} - \varepsilon\widetilde{u}_1, \widetilde{n}+1)}{1 + \varepsilon u'(\widetilde{x} - \varepsilon\widetilde{u}_1, \widetilde{n}+1)}$$

(2) The geometric meaning

- The differential structure.
  - Fix n, the jet bundle structure for each slice $\mathcal{T}_n = \mathbb{R} \times \{n\} \times \mathbb{R}$:

$$J^\infty(\mathcal{T}_n) = (u, u', u'', \ldots)$$

  - The total jet space is

$$J^\infty(\mathcal{T}) \cong \mathbb{Z} \times J^\infty(\mathcal{T}_n)$$

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- The difference structure [Mansfield–Rojo-Echeburúa–Hydon–P, 2019].
  - The total space $\mathcal{T} = \mathbb{R} \times \mathbb{Z} \times \mathbb{R}$ is preserved by all translations

$$T_k : \mathcal{T} \to \mathcal{T}, \quad T_k : (x, n, u) \mapsto (x, n+k, u)$$

  - Prolongation space over n, denoted by $P(\mathcal{T}_n)$, is obtained by pulling back the value of u at each $\mathcal{T}_{n+k}$ by using T_k:

$$u_k = T_k^*(u|_{\mathcal{T}_{n+k}})$$

- The DD structure.
  - Extend the translations T_k to the total jet space $J^\infty(\mathcal{T})$:

$$\begin{aligned} T_k : J^\infty(\mathcal{T}) &\to J^\infty(\mathcal{T}) \\ (x, n, \ldots, u^{(j)}, \ldots) &\mapsto (x, n+k, \ldots, u^{(j)}, \ldots) \end{aligned}$$

  - Pulling back values of jets over $n+k$ to n gives the space $P(J^\infty(\mathcal{T}_n))$. The total prolongation space is

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Remark. Let f be a function on $P(J^\infty(\mathcal{T}))$, locally expressed as

$$f_n = f(x, n, \ldots, u_l^{(j)}, \ldots).$$

The pull back of $f_{n+k} = f(x, n+k, \ldots, u_l^{(j)}, \ldots)$ using T_k gives

$$T_k^* f_{n+k} = f(x, n+k, \ldots, u_{l+k}^{(j)}, \ldots),$$

which is defined as the shift of f_n, i.e.,

$$S^k f_n := T_k^* f_{n+k}.$$

Regular transformations

Definition. Transformations $\mathbf{v} = \xi\partial_x + \phi\partial_u$ satisfying $S\xi = \xi$, meaning $\xi = \xi(x)$, are called *regular/intrinsic*.

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Theorem. [P–Hydon, 2021] A one-parameter local Lie group of transformations

$$\Gamma_\varepsilon : \mathcal{T} \to \mathcal{T}$$

preserves the geometric structure of the total prolongation space $P(J^\infty(\mathcal{T}))$ if and only if it is a group of regular transformations.

Prolongation of vector fields

Theorem. [P–Hydon, 2021] Let $\mathbf{v} = \xi(x,n,u)\partial_x + \phi(x,n,u)\partial_u$ be the infinitesimal generator of a local Lie group of transformations

$$\Gamma_\varepsilon : (x,n,u) \mapsto (\widetilde{x}, n, \widetilde{u}),$$

where $\Gamma_0 = \mathrm{id}$ and

$$\xi = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\widetilde{x}, \quad \phi = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\widetilde{x}.$$

Its prolongation to higher jets are given by the evolutionary representative

$$\mathbf{pr}\mathbf{v} = \xi D + Q\partial_u + (DQ)\partial_{u'} + (SQ)\partial_{u_1} + (DSQ)\partial_{u_1'} + \cdots$$

where $Q(x,n,u,u') = \phi - \xi u'$ is the corresponding characteristic.

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Remark. Symmetries of a DDE $F = 0$ can then be computed through the **linearized symmetry condition** (equivalent to the Taylor expansion approach):

$$\mathbf{pr}\mathbf{v}(F) = 0 \text{ whenever } F = 0.$$

The Toda lattice

$$u'' = \exp(u_{-1} - u) - \exp(u - u_1)$$

- All of its Lie point symmetries are

$$x\partial_x + 2n\partial_u, \quad \partial_x, \quad x\partial_u, \quad \partial_u$$

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- Compared with [Levi–Winternitz, 1991]:

$$x\partial_x + 2n\partial_u, \quad f(n)\partial_x, \quad x\partial_u, \quad \partial_u$$

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  where f is arbitrary

Remark. $f(n)\partial_x$ ($f \neq$ const.) is not a symmetry of the Toda lattice.

Partitioned DDEs

Example. The simple DDE

$$u' = \frac{u_2}{u}$$

admits symmetries (using the **linearized symmetry condition** or Taylor expansion)

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = (-1)^n \partial_x, \quad \mathbf{v}_3 = (-1)^n \left(x\partial_x + u\partial_u \right),$$
$$\mathbf{v}_4 = x\partial_x + u\partial_u, \quad \mathbf{v}_5 = 2^{\lfloor \frac{n}{2} \rfloor} u\partial_u, \quad \mathbf{v}_6 = (-1)^n 2^{\lfloor \frac{n}{2} \rfloor} u\partial_u,$$

where $\lfloor \cdot \rfloor$ denotes the floor function, e.g., $\left\lfloor \frac{n}{2} \right\rfloor$ meaning the greatest integer less than or equal to $n/2$.

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where $\lfloor \cdot \rfloor$ denotes the floor function, e.g., $\left\lfloor \frac{n}{2} \right\rfloor$ meaning the greatest integer less than or equal to $n/2$.

Remark. A DDE can admit non-regular symmetries only when it is a partitioned equation of the form

$$F(x, n, (u, u', \ldots), (u_K, u'_K, \ldots), (u_{2K}, u'_{2K}, \ldots), \ldots) = 0,$$

where the integer is $K \geq 2$ (or $K \leq -2$ for a backward DDE).

Group-invariant solutions/Similarity reduction: Toda

$$u'' = \exp(u_{-1} - u) - \exp(u - u_1)$$

- Recall its symmetries:

$$\mathbf{v}_1 = x\partial_x + 2n\partial_u, \quad \mathbf{v}_2 = \partial_x, \quad \mathbf{v}_3 = x\partial_u, \quad \mathbf{v}_4 = \partial_u$$

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- $\mathbf{v}_1 + C_0\mathbf{v}_4$: The invariants are n and $\frac{u}{2n+C_0} - \ln x$.

$$u(x,n) = (2n + C_0)\ln x - \sum_{k=0}^{n} \ln\left(k^2 + (C_0+1)k + C_1\right) + C_2$$

- $\mathbf{v}_2 + C_0\mathbf{v}_3$: The invariants are n and $u - \frac{C_0 x^2}{2}$.

$$u(x,n) = \frac{C_0}{2}x^2 - \sum_{k=0}^{n} \ln\left(-C_0 k + C_1\right) + C_2$$

Group-invariant solutions/Similarity reduction: Volterra

The Volterra equation

$$u' = u(u_1 - u_{-1})$$

- All (Lie point) symmetries:

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = -x\partial_x + u\partial_u.$$

Group-invariant solutions/Similarity reduction: Volterra

The Volterra equation

$$u' = u(u_1 - u_{-1})$$

- All (Lie point) symmetries:

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = -x\partial_x + u\partial_u.$$

- Invariants of $\mathbf{v} = C_0\mathbf{v}_1 + \mathbf{v}_2$ are n and $(x - C_0)u$:

$$u(x, n) = \frac{C_1 + C_2(-1)^n - n}{2(x - C_0)},$$

where C_0, C_1, C_2 are all arbitrary constants.

DD variational calculus

Theorem. A DD variational problem

$$\sum_{n=0}^{N} \int_{\Omega} L(x, n, u, u_1, u', \ldots)\, \mathrm{d}x,$$

with Ω open and connected, is invariant with respect to the vector field $\mathbf{v} = \xi \partial_x + \phi \partial_u$ if and only if there exist functions P^x and P^n such that the Lagrangian satisfies the *criterion of variational invariance*:

$$\mathbf{pr}\mathbf{v}(L) + L(D\xi) = DP^x + (S - \mathrm{id})P^n.$$

DD variational calculus

Theorem. A DD variational problem

$$\sum_{n=0}^{N} \int_{\Omega} L(x, n, u, u_1, u', \dots) \, \mathrm{d}x,$$

with Ω open and connected, is invariant with respect to the vector field $\mathbf{v} = \xi \partial_x + \phi \partial_u$ if and only if there exist functions P^x and P^n such that the Lagrangian satisfies the *criterion of variational invariance*:

$$\mathbf{prv}(L) + L(D\xi) = DP^x + (S - \mathrm{id})P^n.$$

- A DD Lagrangian $L(x, n, u, u_1, u', \dots)$
- DD Euler–Lagrange equation: $\mathbf{E}(L) = 0$ with DD Euler operator

$$\mathbf{E} := \sum_{j,l} (-D)^j S^{-l} \frac{\partial}{\partial u_l^{(j)}}, \quad u_l^{(j)} = D^j S^l u$$

- Conservation law: $DP^x + (S - \mathrm{id})P^n = Q\mathbf{E}(L)$ where Q is called a characteristic

Noether's Theorem for DDEs

Noether's Theorem. There is a one-to-one correspondence between symmetry characteristics of a variational problem with Lagrangian L and characteristics of conservation laws of the corresponding Euler–Lagrange equations.

$$\mathbf{prv}(L) + L(D\xi) = DP^x + (S - \mathrm{id})P^n$$

$$\text{where } \mathbf{prv} = \xi D + Q\partial_u + (DQ)\partial_{u'} + \cdots$$

$$\Leftrightarrow$$

$$DA^x + (S - \mathrm{id})A^n = Q\mathbf{E}(L)$$

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Remark. All results can be generalised to higher-order symmetries:

$$\text{Lie point symmetries } Q = \phi(x, n, u) - \xi(x, n, u)u'$$
$$\Rightarrow$$
$$\text{higher-order symmetries } Q(x, n, [u])$$

$\dagger [u] = (u, u_1, u', \ldots)$ is a shorthand for u and finitely many of its shifts and derivatives.

Volterra equation $u' = u(u_1 - u_{-1})$

- By a change of variables

$$u = \exp(v_1 - v_{-1}),$$

the Volterra equation becomes the Euler–Lagrange equation of

$$L = v_{-1}v' + \exp(v_1 - v_{-1}).$$

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- Variational symmetries $\mathbf{v} = (C_1 + (-1)^n C_2)\,\partial_v \Leftrightarrow$ conservation laws

$$D\,(\ln u) + (S - \mathrm{id})\,(-u - u_{-1}) = 0,$$
$$D\,((-1)^n \ln u) + (S - \mathrm{id})\,((-1)^n\,(u - u_{-1})) = 0.$$

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Remark. A general *inverse theory* is not yet available.

Noether's Second Theorem

Noether's Second Theorem. A DD variational problem admits symmetries whose characteristic $Q(x, n, [u; f])$ depends on R independent arbitrary functions

$$\big(f^1(x,n), f^2(x,n), \ldots, f^R(x,n)\big)$$

and their derivatives and shifts *if and only if* there exist DD operators $\mathcal{D}_r^\alpha$ (not all zero) yielding R independent DD relations among the Euler–Lagrange equations:

$$\mathcal{D}_r^\alpha \mathbf{E}_\alpha(L) \equiv 0, \quad r = 1, 2, \ldots, R.$$

Gauge-symmetry preserving semi-discretisations: An example

Interaction of a scalar particle of mass m and charge e with an electromagnetic field:

- Space-time coordinated by $(x^0 = t, x^1, x^2, x^3)$ $(x^0 = n$ in the DD case)
- Dependent variables:
  - scalar and complex-valued ψ: wavefunction
  - real-valued A^μ: electromagnetic four-potential
- Metric $\eta = \mathrm{diag}(-1, 1, 1, 1)$

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The continuous system:

- The Lagrangian:

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\nabla_\mu \psi)(\nabla_\mu \psi)^* + m^2 \psi \psi^*$$

  where

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}, \quad \nabla_\mu = D_\mu + \mathrm{i} e A_\mu$$

- Euler–Lagrange equations:

$$\mathbf{E}_\psi(L) = 0, \quad \mathbf{E}_{\psi^*}(L) = 0, \quad \mathbf{E}_{A^\mu}(L) = 0$$

- Gauge-symmetries:

$$\psi \mapsto \exp(-\mathrm{i} e \lambda), \quad A^\mu \mapsto A^\mu + \eta^{\mu\nu} \lambda_{,\nu}$$

  where the function $\lambda(x^0, x^1, x^2, x^3)$ is arbitrary and real-valued.

- Differential relation of Euler–Lagrange equations:

$$-\mathrm{i} e \psi \mathbf{E}_\psi(L) + \mathrm{i} e \psi^* \mathbf{E}_{\psi^*}(L) - D_\mu \left(\eta^{\nu\mu} \mathbf{E}_{A^\nu}(L) \right) \equiv 0$$

Fully discrete counterpart: [Christiansen–Halvorsen, 2011] (see also [Hydon–Mansfield, 2011])

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A DD counterpart: time t is discretized with time step h.

- The DD Lagrangian:

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\nabla_\mu \psi)(\nabla_\mu \psi)^* + m^2 \psi \psi^*$$

where by denoting the forward difference operator $\Delta = \frac{S - \mathrm{id}}{h}$,

$$\begin{aligned} F_{\mu\nu} &= -F_{\nu\mu}, \quad \forall \mu, \nu, \\ F_{0\mu} &= \Delta A_\mu - D_\mu A_0, \quad \mu \neq 0, \\ F_{\mu\nu} &= A_{\mu,\nu} - A_{\nu,\mu}, \quad \mu \neq 0, \nu \neq 0 \end{aligned}$$

and

$$\begin{aligned} \nabla_0 &= \Delta + \frac{1 - \exp(-\mathrm{i} e h A_0)}{h}, \\ \nabla_\mu &= D_\mu + \mathrm{i} e A_\mu, \quad \mu \neq 0. \end{aligned}$$

- DD Euler–Lagrange equations:

$$\mathbf{E}_{\psi}(L) = 0, \quad \mathbf{E}_{\psi^*}(L) = 0, \quad \mathbf{E}_{A^\mu}(L) = 0$$

- Gauge-symmetries:

$$\psi \mapsto \exp(-\mathrm{i}e\lambda), \quad A^0 \mapsto A^0 - \Delta\lambda, \quad A^\mu \mapsto A^\mu + \sum_{\nu=1}^{3} \eta^{\mu\nu}\lambda_{,\nu} \;\; (\mu \neq 0)$$

where the function $\lambda(n, x^1, x^2, x^3)$ is again arbitrary and real-valued.

- Differential relation of Euler–Lagrange equations:

$$-\mathrm{i}e\psi\mathbf{E}_{\psi}(L) + \mathrm{i}e\psi^*\mathbf{E}_{\psi^*}(L) - \Delta^\dagger(\mathbf{E}_{A^0}(L)) - \sum_{\mu,\nu=1}^{3} D_\mu\left(\eta^{\nu\mu}\mathbf{E}_{A^\nu}(L)\right) \equiv 0$$

where $\Delta^\dagger$ is adjoint to Δ:

$$\Delta^\dagger = -\frac{\mathrm{id} - S^{-1}}{h}.$$

Summary

- The general prolongation formulation for symmetries of DDEs is proved analytically, that allows us to compute symmetries systematically.
- Continuous symmetries can be used to construct group-invariant solutions of DDEs.
- Noether's two theorems are extended to DD variational problems.
  - [1] Finite-dimensional variational symmetries and conservation laws
  - [2] Infinite-dimensional variational symmetries and differential relations of (under-determined) Euler–Lagrange equations
  - [1.5] An intermediate theorem (infinite-dimensional variational symmetries that are subject to constraints)

Thanks a lot for your attention.

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