Representing graphs with sublinear separators

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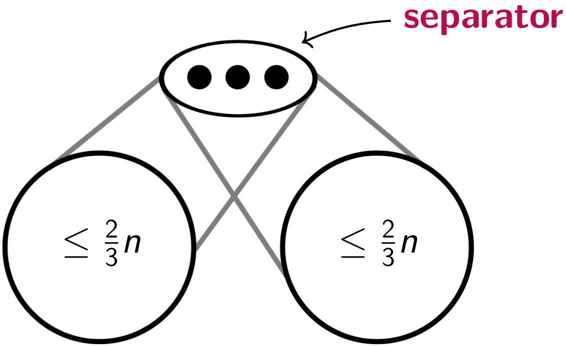
Joint work with Zdeněk Dvořák and Sergey Norin

November 24th, 2021

Banff, Graph Product Structure Theory


separator
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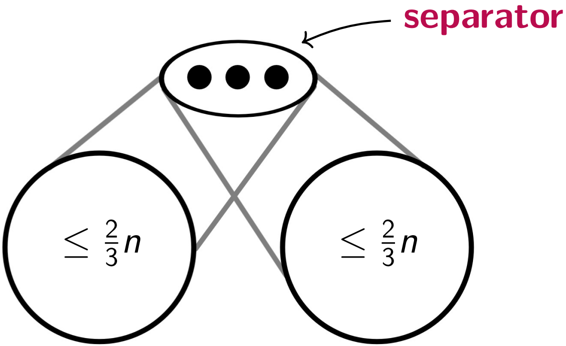
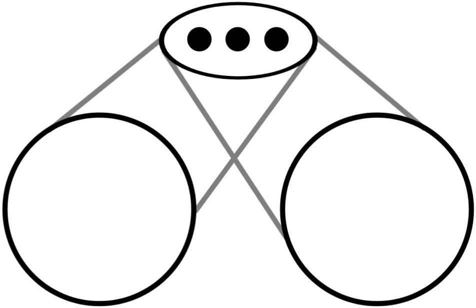


A class has **sublinear separators** if its n-vertex subgraphs have separators of size $\mathcal{O}(n^{1-\epsilon})$ for $\epsilon > 0$.

Can we describe the **structure** of these classes?



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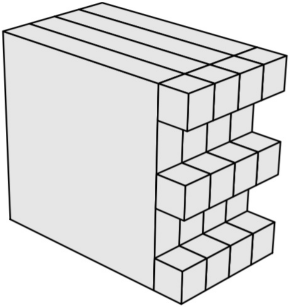

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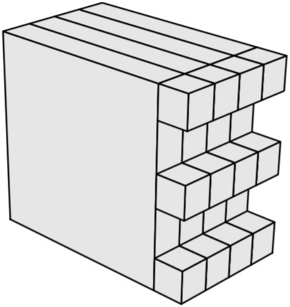

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- is general
- respects product structure
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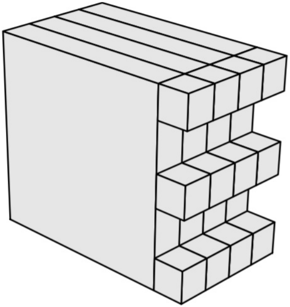

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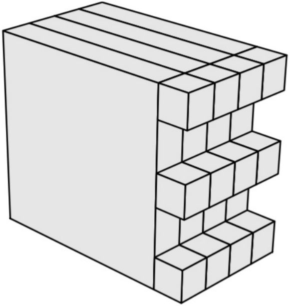

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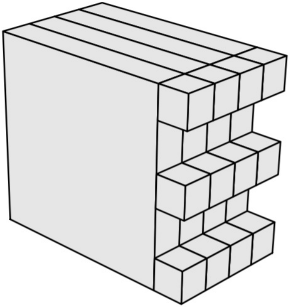

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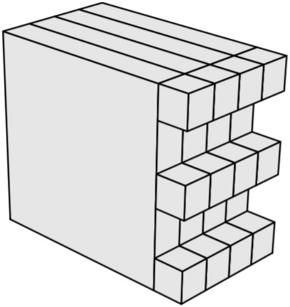
generalized coloring numbers



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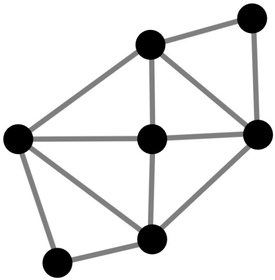

(Lipton-Tarjan 79)

The following classes have **sublinear separators**.

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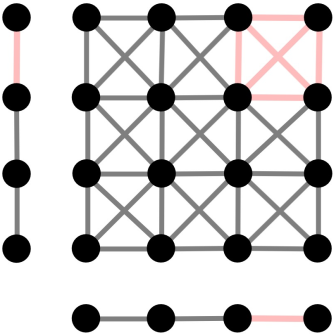

(Dvořák-Huynh-Joret-Liu-Wood 21)

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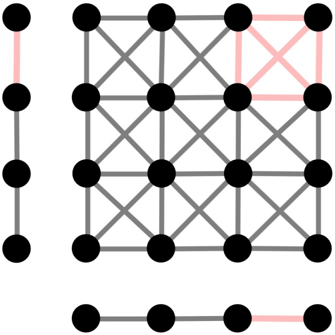
  (for each c, subgraphs of $H \boxtimes P$ where $\mathrm{tw}(H) \leq c$)



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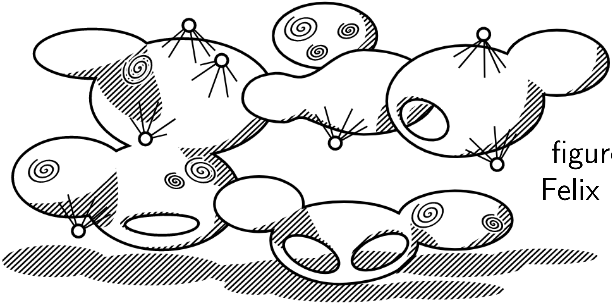

figure by Felix Reidl

(Alon-Seymour-Thomas 90)

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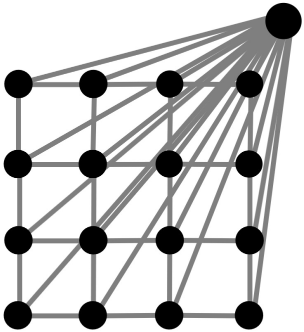

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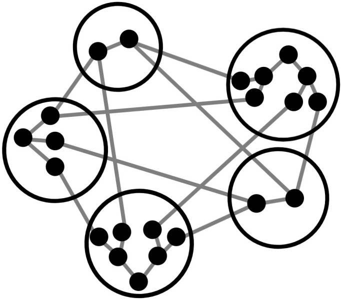

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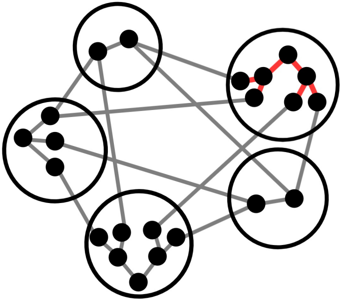

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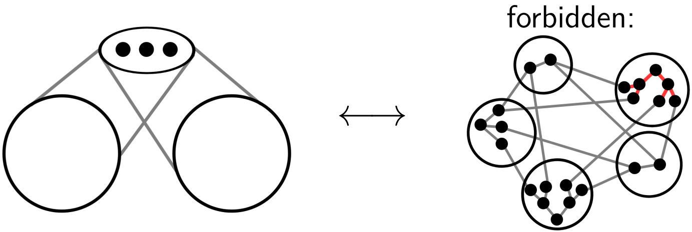

These are the **only** classes with sublinear separators.

(Dvořák-Norin 16)

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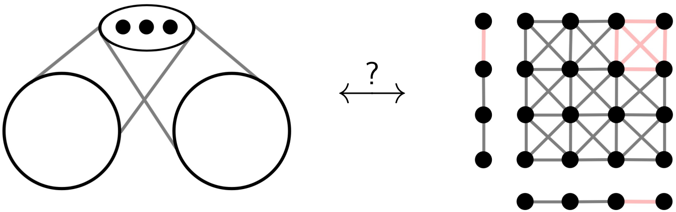

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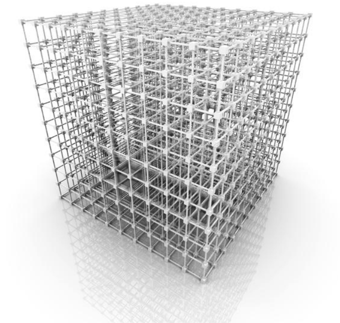
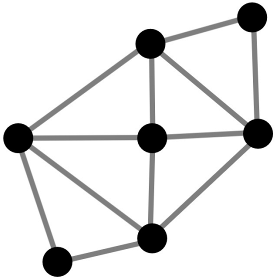
Subgraphs of $H \boxtimes \overbrace{P \boxtimes \ldots \boxtimes P}^{d}$: $\mathcal{O}(n^{1-\frac{1}{d+1}})$



(Dvořák-Huynh-Joret-Liu-Wood 21)

There is also a d-dimensional analog of planarity...



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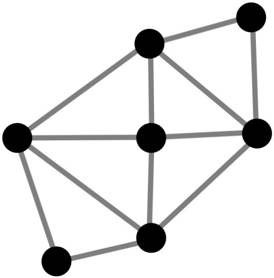

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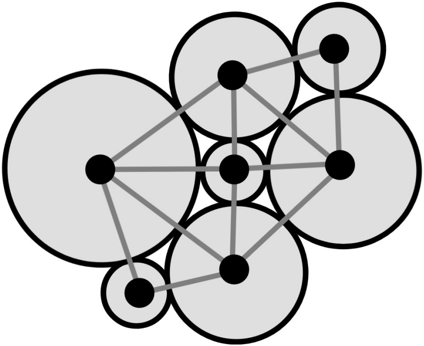

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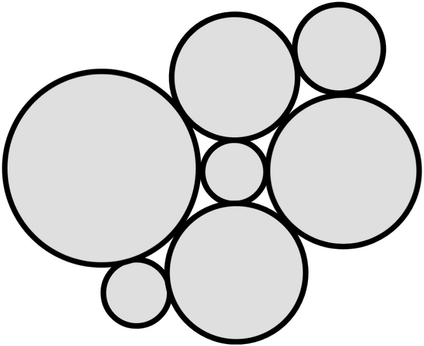

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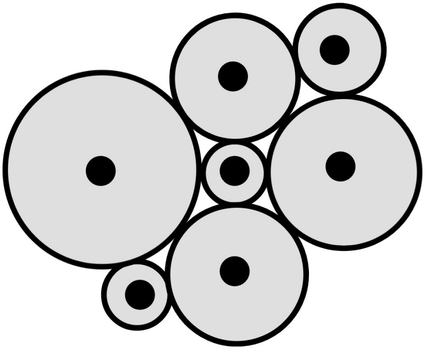

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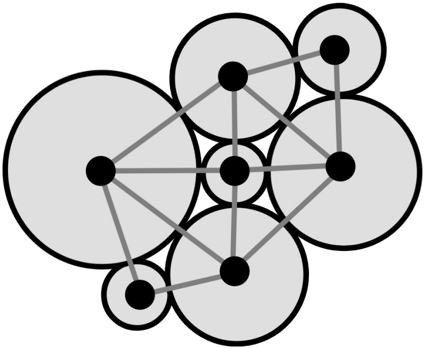

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The **intersection graphs** of internally disjoint balls in $\mathbb{R}^d$ have separators of size $\mathcal{O}(n^{1-\frac{1}{d}})$.
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The **intersection graphs** of ***k*-wise disjoint** balls in $\mathbb{R}^d$ have separators of size $\mathcal{O}(n^{1-\frac{1}{d}})$.
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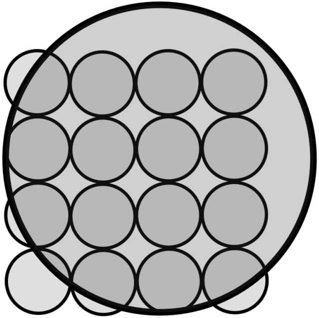

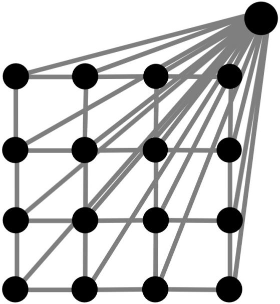

The **intersection graphs** of k-**wise disjoint** balls in $\mathbb{R}^d$ have separators of size $\mathcal{O}(n^{1-\frac{1}{d}})$.
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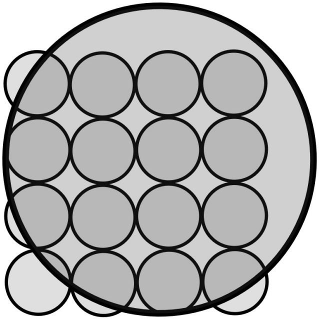

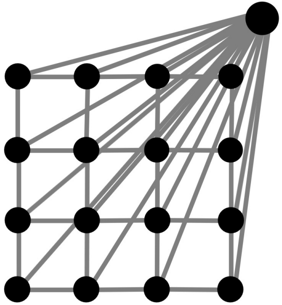

The same holds if instead of balls, we consider compact convex sets of **bounded aspect ratio**.

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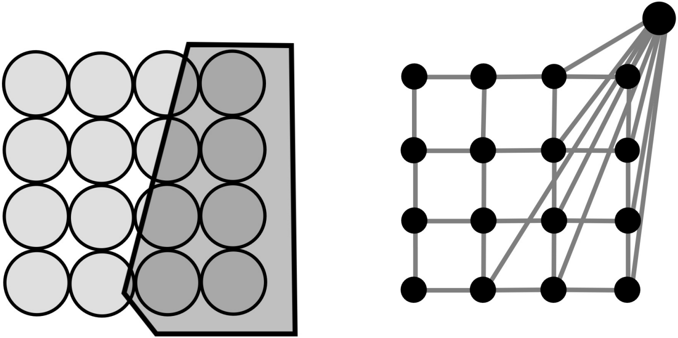

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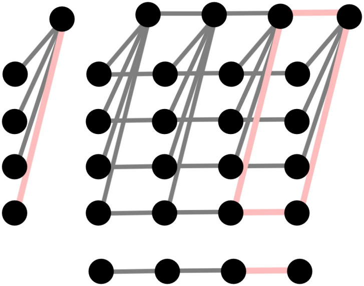
For any d, k, r, there exists Star$\Box$P which is not a subgraph of an intersection graph of k-wise disjoint compact convex sets in $\mathbb{R}^d$ with aspect ratio $\leq r$.



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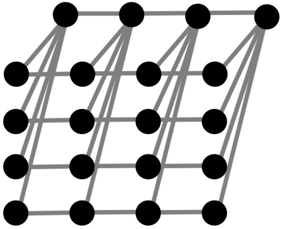

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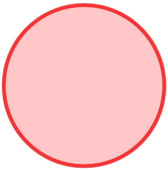

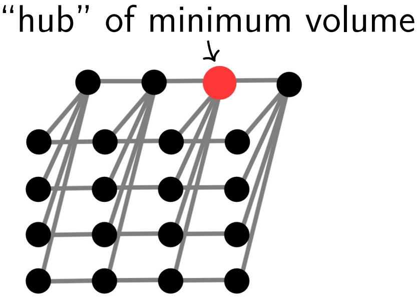

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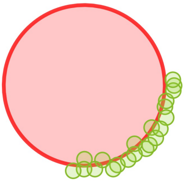

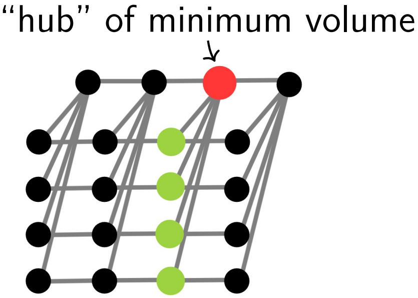

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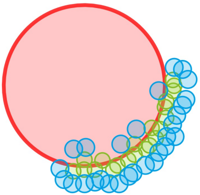

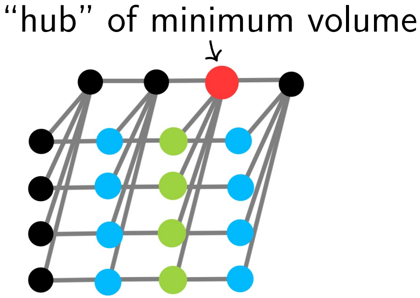

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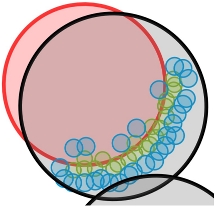

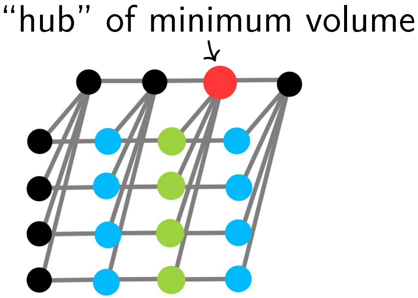

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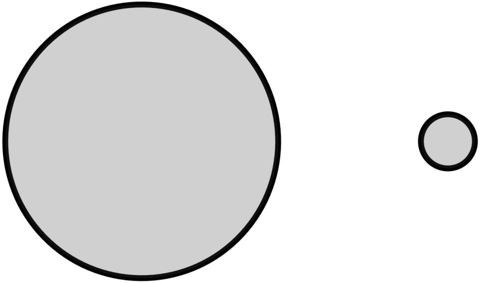
Instead we have a property about pairs of shapes.



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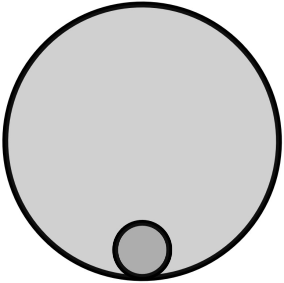

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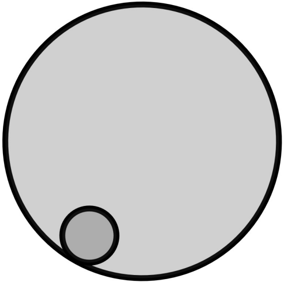

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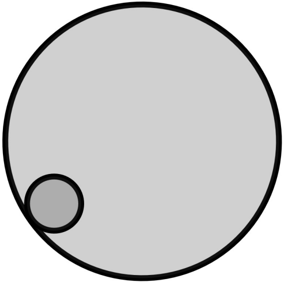

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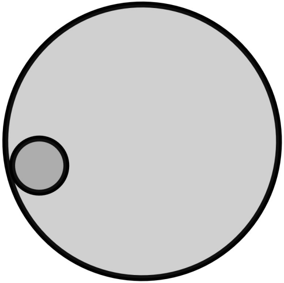

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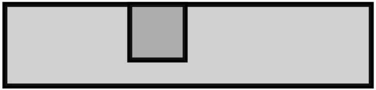

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For each point x in the "larger" shape, there is a translate S of the "smaller" shape which contains x

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r-**comparable**

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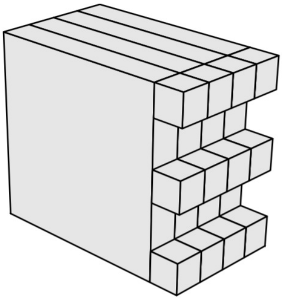
Theorem

*The class of intersection graphs of r-**comparable**, k-wise disjoint, compact convex sets in $\mathbb{R}^d$ has separators of size $\mathcal{O}(n^{1-\frac{1}{2d+4}})$.*



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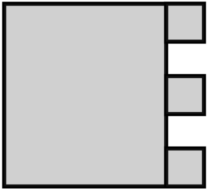

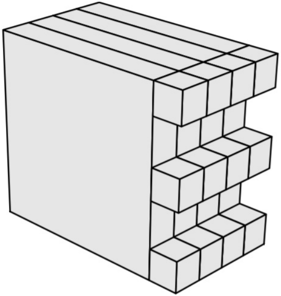

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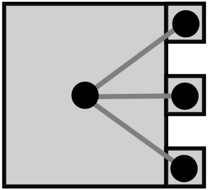

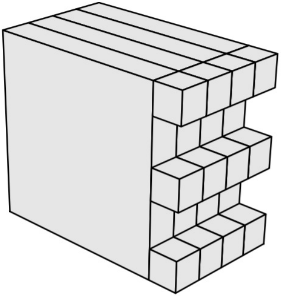

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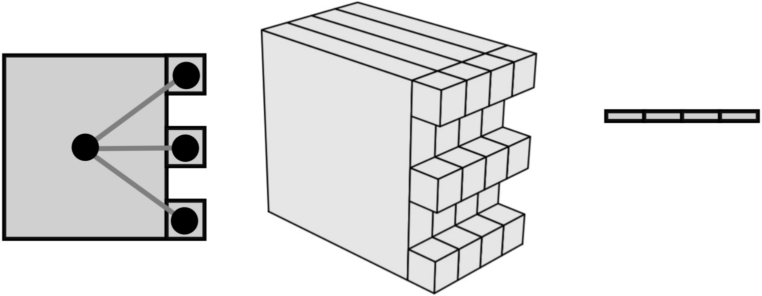

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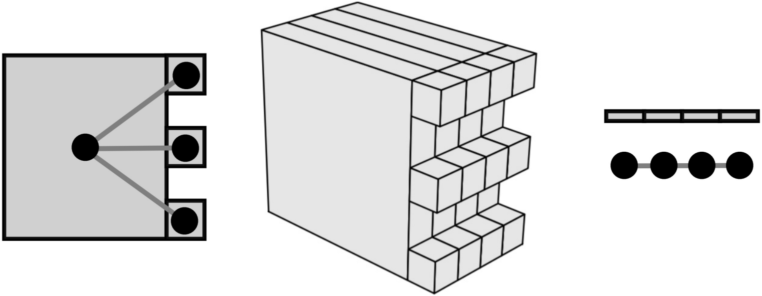

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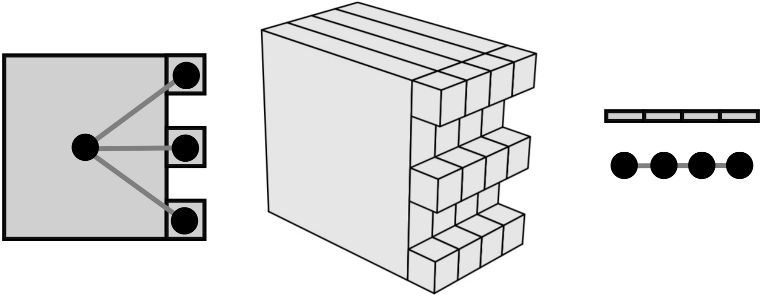

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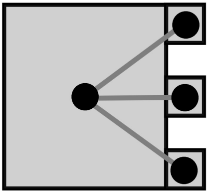
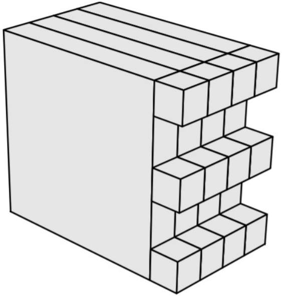
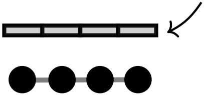




Products work if all of these shapes are both "large" and "small".



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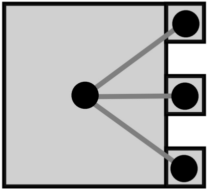

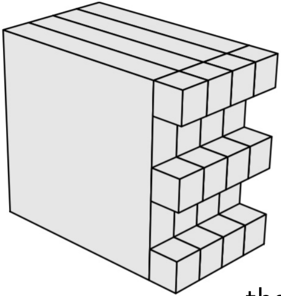

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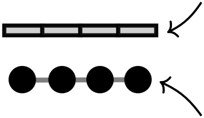

Using a theorem of Krauthgamer-Lee 07, these are exactly subgraphs of $P \boxtimes \ldots \boxtimes P$.

Lemma (see Dvořák 18 and Esperet-Raymond 18)

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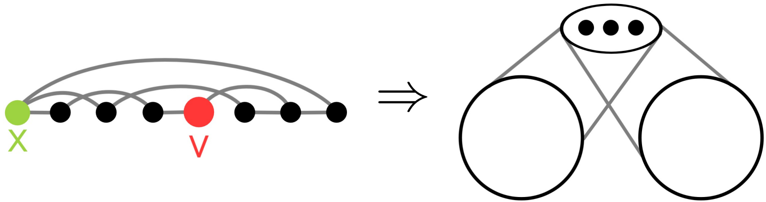

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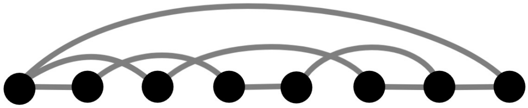

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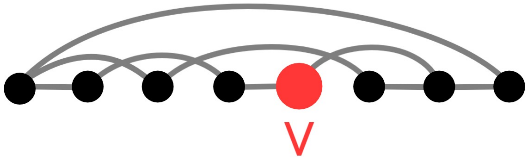

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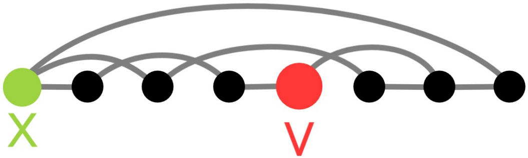

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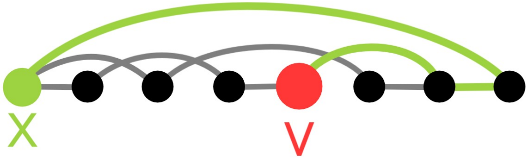

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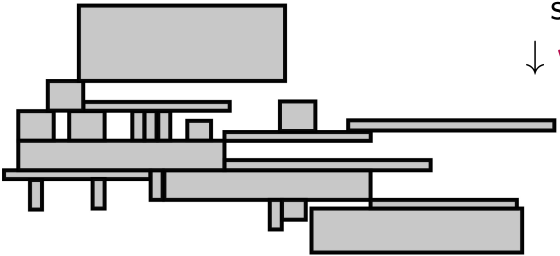

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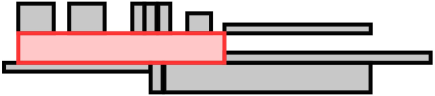

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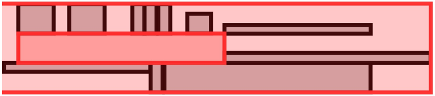

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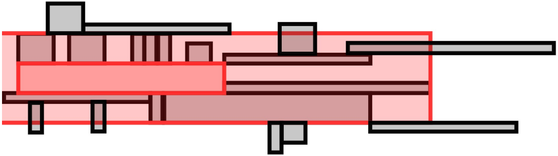

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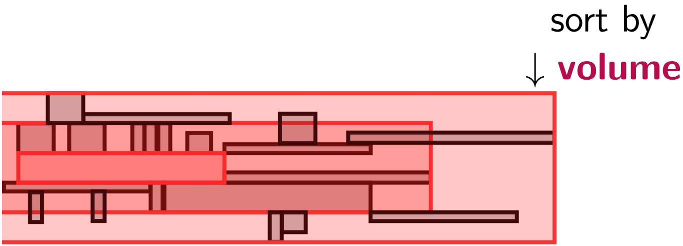

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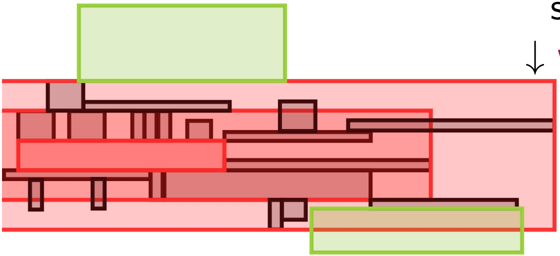

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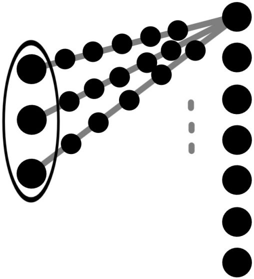

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Problem (van den Heuvel-Kierstead 19)

If a class has **strong coloring numbers** $\leq poly(r)$, *then is there a* **single** *vertex ordering for all* r?

Thank you!