p-adic limit of (relative) orbital integral

joint D Disegni (work in progress)

I) Application to p-adic BBK conj.

II) A p-adic AGGP conj

III) Proof - outline

I) (p-adic) B-B-K

X/F = number field, (smooth proj), p: good prime

$$\begin{array}{ccccccc} 0 & \longrightarrow & Ch^*(X)_0 & \longrightarrow & Ch^*(X) & \longrightarrow & H^{2*}(X_{\bar F}, \mathbb{Q}_p) \\ & & \big\downarrow {\scriptstyle \text{p-adic Abel-Jacobi}} & & & & \\ & & H^1_f(F, H^{2*-1}(X_{\bar F}, \mathbb{Q}_p)) & & & & \end{array}$$

$$0 \to H^{2*-1} \to \;?\; \to \mathbb{Q}_p \to 0$$

Bloch-Kato selmer group

$L(s, H^*(X))$: Hasse-Weil L-fun.

Conj (Beilinson Bloch Kato) (p-adic version)

$$\operatorname{ord}_{s=\text{center}} L_p(s, H^{2i-1}(X)) = \dim H^1_f(F, H^{2i-1}(X)(i))$$

p-adic L-function at p ordinary.

$L(s=\text{center}, H^{2i-1} \otimes \chi)$

(χ: finite order character)

(Coates-Perrin-Riou)

Sample Thm:

A_1, A_2 elliptic $/\mathbb{Q}$ (semistable)

F/F_0 CM extn (solvable $/\mathbb{Q}$)

$X = (A_1^{n-1} \times A_2^n)_F \qquad n \geq 1.$

$\mathrm{Gal}(\bar{F}/F) \circlearrowright V = \mathrm{Sym}^{n-1} H^1(A_1) \otimes \mathrm{Sym}^n H^1(A_2) \subseteq H^{2n+1}(X)$

(Newton-Thorne: automorphic)

Theorem * Let p ordinary $\gg 0$ + mild ramification

If $\mathrm{ord}\, L_p(s, V) \leq 1$, then:

$$\mathrm{ord}\, L_p(s, V) = \mathrm{rk}\, H^1_f(F, V(*))$$

Rmk: $\mathrm{ord}\, L_p = 0$ by Liu-Tian-Xiao-Zhang-Zhu

II) p-adic AGGP conj: F/F_0 CM extn

$H \subset G$

$W_n \hookrightarrow W_{n+1}$ Hermitian space

dim n $n+1$

$H = U(W_n) \hookrightarrow G = U(W_n) \times U(W_{n+1})$ $/\mathbb{Q}$

sgn $W_n = (n-1, 1), (n,0)$ $\ldots$, $(n,0)$

$W_{n+1} = (n, 1)$ $(n+1, 0)$ $\longrightarrow$

Arith diagonal: $Sh_H \longrightarrow Sh_G$ $/F$

dim: $n-1$ $n + n-1 = 2n-1$

"Working Hypothesis"

$$H^{2n-1}(Sh_G) \doteq \bigoplus_{\pi} \pi \boxtimes V_{\pi^\vee}$$

$\circlearrowleft$

$G(\mathbb{A}_f) \times Gal(\bar{F}/F)$, π tempered cusp.

$$[Sh_H] \in H^1_f(F, H^{2n-1}(-)) = \bigoplus_{\pi} \pi \otimes H^1_f(F, V_{\pi^\vee})$$

p-adic Height pairing (Nekovar).

$$\langle \quad \rangle_p : H^1_f(F\ V_{\pi^\vee}) \times H^1_f(F\ V_\pi) \longrightarrow \mathbb{Q}_p$$

p ordinary

<u>Conj</u> (padic AGGP) $\varphi \in \pi$ $\quad \varphi^\vee \in \pi^\vee$

$$\langle \varphi.[Sh_H], \varphi^\vee[Sh_H] \rangle_p \doteq L_p'(\tfrac{1}{2}, \pi) \cdot \prod_{v \neq \infty} \alpha(\varphi_v, \varphi_v^\vee)$$

$$\varphi = \otimes_v \varphi_v \qquad \varphi^\vee = \otimes_v \varphi_v^\vee$$

<u>Rmk</u>:

i) α_v: local terms: (Ichino-Ikeda, Waldspurger)

ii) <u>padic L function</u>

$$G = U(W_n) \times U(W_{n+1})$$
$$\pi = \pi_n \boxtimes \pi_{n+1}$$

Base change $G' = GL_n, F \times GL_{n+1}, F$

$BC(\pi)$

$$L(s, BC(\pi_n) \times BC(\pi_{n+1}))$$

<u>Theorem</u> Conj above holds if

i) all places $v|p$ split in F/F_0

ii) F/F_0 unramified at all $v \neq \infty$.

iii) π_v unram (or almost unramified) at all inert v

$$\pi = \otimes_v \pi_v$$

III) <u>Outline of proof</u> : RTFs

p-adic L-function $\longleftrightarrow$ p-adic Height

i) <u>Incorporating p-adic L-function into RTF.</u>

<u>p-adic R-S</u>

$\Pi = \pi_n \boxtimes \pi_{n+1}$

$\varphi \otimes \varphi' \in \Pi^{I}$

$$\lambda_{RS}(\varphi\otimes\varphi', \chi) = \int_{[H]} \varphi(h)\, \varphi'\begin{pmatrix} h & \\ & 1 \end{pmatrix} |h|^{s-\frac{1}{2}} \chi(h)\, dh \sim L(\pi_n \times \pi_{n+1} \otimes \chi, s)$$

$s = \frac{1}{2}$

$\{\chi : \mathbb{A}^\times / F^\times \xrightarrow[\text{fin order}]{} \mathbb{C}^\times\}$

$$\chi \longmapsto \lambda_{RS}(\text{"}u_p^{-m}\text{"} * \gamma_m(\varphi\otimes\varphi'), \chi)$$

$c(\chi)$ = conductor at p

$m > c(\chi)$

$\frac{1}{\alpha^m}$

$$u_p \in \mathcal{H}(I \backslash G / I)$$

$\alpha = u_p$ - eigenvalue

$$u_p = \mathbb{1}_{I_n t_n I_n} \otimes \underline{\qquad\qquad}$$

$$t_n = \begin{pmatrix} p^{n-1} & & & \\ & \ddots & & \\ & & p & \\ & & & 1 \end{pmatrix}$$

$$\gamma_m = \left(1_n, \left(\begin{array}{ccc|c} 1 & & & 1 \\ & 1 & & \vdots \\ & & 1 & 1 \\ \hline & & & 1 \end{array}\right)\right)(t_n^m, t_{n+1}^m)$$

RTF (Jacquet–Rallis)

$$\sum_{\pi} I_{\pi}(f) = \sum_{\gamma} \mathrm{Orb}(\gamma, f)$$

$$f = f^{p} \otimes f_p$$

$$f_{p,N,m} \approx (p\text{-power}) \cdot \gamma_m \cdot u_p^{N!-m} \quad \text{"}\infty\text{"} \quad N \to \infty$$

Thm (p-adic limit of orbital integral)

If v/p split in F/F_0, then

$$\mathrm{Orb}(\gamma, f_{p,N,m}, \chi) \xrightarrow[m > c(\chi)]{N \to \infty} \begin{cases} 0 \\ \text{constant} \end{cases} \quad (p\text{-adically})$$

$$\gamma \in (H_1 \backslash G' / H_2)_{reg}.$$

Question: v nonsplit ?

Geometric side (p-adic Height)

$y_1, y_2 \in Ch^*(Sh_G), \quad y_1 \cap y_2 = \phi.$

$$\langle y_1 \ \ y_2 \rangle_p = \sum_v \langle y_1 \ \ y_2 \rangle_v .$$

$v \nmid p$

$$\langle y_1, y_2 \rangle_v = \chi\left(\widetilde{Sh_G}, \ \mathcal{O}_{\widetilde{y_1}} \overset{\mathbb{L}}{\otimes} \mathcal{O}_{\widetilde{y_2}}\right) \log_p q_v$$

reg. int. model

$\cap$

$\mathbb{Q}_p$

v Hyperspecial: Arithmetic F. L.

v almost unramified: Arithmetic transfer (R-S-Z + Zhiyu Zhang Thesis)

$v \mid p$: (v split in F/F_0)

Apply Hida ordinary projector $\mathcal{U}_p^{N!}$

$N \longrightarrow \infty$

(Perrin-Riou GL_2)