Multipliers and quasicuspidal convolution operators

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Basic Functions, Orbital Integrals and Beyond Endoscopy
Banff, November 18th 2021

Lindenstrauss-Venkatesh ('05) : soft proof of the existence of spherical cusp forms e.g. even Maass forms.

- Set $\Gamma = \mathrm{PGL}_2(\mathbb{Z}) \curvearrowright \mathbb{H} = \mathrm{G}/\mathrm{K}$: upper half plane with $\mathrm{G} = \mathrm{PGL}_2(\mathbb{R})$, $\mathrm{K} = \mathrm{PO}_2(\mathbb{R})$.
- Then, $\mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K})$ acts on $\mathrm{L}^2(\Gamma\backslash\mathbb{H})$ (by right convolution) and we have :

$$\mathrm{L}^2(\Gamma\backslash\mathbb{H}) = \mathrm{L}^2_{cusp} \oplus \mathbb{C}\mathbf{1} \oplus \int_{\mathbb{R}}^{\oplus} \mathbb{C}\,\mathrm{E}_{1/2+it}\,dt.$$

- For $k \in \mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K})$, $\mathrm{R}(\mathrm{k})\mathrm{E}_{1/2+\lambda} = \widehat{\mathrm{k}}(\lambda)\mathrm{E}_{1/2+\lambda}$ where $\lambda \mapsto \widehat{\mathrm{k}}(\lambda)$ is the spherical transform of k.
- Paley-Wiener thm (Helgason) : $\mathrm{k} \mapsto \widehat{\mathrm{k}}$ induces $\mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K}) \simeq \mathrm{PW}_{\text{even}}(\mathbb{C})$ (where $\mathrm{PW}_{\text{even}}(\mathbb{C}) := \mathcal{F}\,\mathrm{C}^\infty_{c,\text{even}}(\mathbb{R})$).
- On the other hand, $\mathrm{T}_p\mathrm{E}_{1/2+\lambda} = (p^\lambda + p^{-\lambda})\mathrm{E}_{1/2+\lambda}$.
- Let $\mathrm{U}_{\log p} : \mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K}) \to \mathrm{C}_c^\infty(\mathrm{K}\backslash\mathrm{G}/\mathrm{K})$ st $(\mathrm{U}_{\log p}\mathrm{k})^\wedge(\lambda) = (p^\lambda + p^{-\lambda})\widehat{\mathrm{k}}(\lambda)$.
- Remark that $\mathrm{R}_{p,\mathrm{k}} := \mathrm{T}_p\mathrm{R}(\mathrm{k}) - \mathrm{R}(\mathrm{U}_{\log p}\mathrm{k})$ kills $(\mathrm{L}^2_{cusp})^\perp$.
- "High in the cusp" $\mathrm{R}(k)$ & $\mathrm{R}(\mathrm{U}_{\log p}k)$ commute with horizontal translations whereas T_p doesn't $\Rightarrow$ We can arrange $\mathrm{R}_{p,k} \neq 0 \Rightarrow$ existence of even Maass forms.

- More generally, L & V construct "many" operators on $\mathrm{L}^2(\Gamma \backslash \mathrm{G}(\mathbb{R}) / \mathrm{K}_\infty)$ with cuspidal image (where : $\mathrm{G} / \mathbb{Q}$ split adjoint, $\Gamma \subset \mathrm{G}(\mathbb{Q})$ congruence subgroup, $\mathrm{K}_\infty \subset \mathrm{G}(\mathbb{R})$ maxl compact) $\rightsquigarrow$ Weyl's law.

However :

- L & V operators always kill some interesting automorphic forms like $\mathsf{Sym}^2 \varphi$ for φ a form on GL_2 ;
- It only works for forms that are spherical at the Archimedean place.

Schwartz spaces

- $\mathrm{G}/\mathbb{Q}$ conn. reductive, $\mathbb{A} = \mathbb{R} \times \prod_p' \mathbb{Q}_p = \mathbb{R} \times \mathbb{A}_f$, $\mathrm{K} = \mathrm{K}_S \times \prod_{p \notin \mathrm{S}} \mathrm{K}_p \subset G(\mathbb{A}_f)$ a "level" with K_p hyperspecial for $p \notin \mathrm{S}$.
- Archimedean Schwartz space :

$$\mathcal{S}(\mathrm{G}(\mathbb{R})) = \{f \in \mathrm{C}^\infty(\mathrm{G}(\mathbb{R})) \mid \forall \mathrm{D} : \text{ polyn. differential op.}, |\mathrm{D} f| \ll 1\}$$
$$= \{f \in \mathrm{C}^\infty(\mathrm{G}(\mathbb{R})) \mid \forall X \in \mathcal{U}(\mathfrak{g}(\mathbb{R})), R > 0 : |R_X f(g)| \ll \|g\|^{-R}\}.$$

It is a space of "very rapidly decreasing" functions (together with all their derivatives) analog to

$$\mathcal{S}_{\exp}(\mathbb{R}) = \{f \in \mathrm{C}^\infty(\mathbb{R}) \mid \forall n \geqslant 0, \mathrm{R} > 0, |f^{(n)}(x)| \ll e^{-\mathrm{R}|x|}\}.$$

The space $\mathcal{S}(\mathrm{G}(\mathbb{R}))$ is an algebra under convolution $*$.

- Global Schwartz space :

$$\mathcal{S}(\mathrm{G}(\mathbb{A}))_{\mathrm{K}} = \mathcal{S}(\mathrm{G}(\mathbb{R})) \otimes \bigotimes_p{}' \mathrm{C}_c(\mathrm{K}_p \backslash \mathrm{G}(\mathbb{Q}_p)/\mathrm{K}_p)$$

that is the space of functions spanned by products $f_\infty \times \prod_p f_p$ where $f_\infty \in \mathcal{S}(\mathrm{G}(\mathbb{R}))$, $f_p \in \mathrm{C}_c(\mathrm{K}_p \backslash \mathrm{G}(\mathbb{Q}_p)/\mathrm{K}_p)$ and $f = \mathbf{1}_{\mathrm{K}_p}$ for a.a. p.

Multipliers

- For $p \notin S$, $\mathcal{H}_p = \mathrm{C}_c(\mathrm{K}_p \backslash \mathrm{G}(\mathbb{Q}_p) / \mathrm{K}_p)$ (spherical Hecke algebra) acts on itself by convolution.
- Multipliers at ∞ : let

$$\mathcal{M}_\infty(\mathrm{G}) = \mathrm{End}_{cont, \mathcal{S}(\mathrm{G}(\mathbb{R}))-bimod}(\mathcal{S}(\mathrm{G}(\mathbb{R})))$$

  be the space of continuous bimodule endomorphisms of $\mathcal{S}(\mathrm{G}(\mathbb{R}))$. It can be identified with the space of "rapidly decreasing invariant distributions on $\mathrm{G}(\mathbb{R})$" acting on $\mathcal{S}(\mathrm{G}(\mathbb{R}))$ by $*$.
- S-multipliers : $\mathcal{M}^{\mathrm{S}}(\mathrm{G}) = \mathcal{M}_\infty(\mathrm{G}) \otimes \bigotimes_{p \notin \mathrm{S}}' \mathcal{H}_p \overset{*}{\curvearrowright} \mathcal{S}(\mathrm{G}(\mathbb{A}))$.

Quasi-cuspidal convolution operators

- Let $\pi = \pi_\infty \otimes \bigotimes'_p \pi_p$ be an irreducible admissible repn of $\mathrm{G}(\mathbb{A})$ st $\pi^{\mathrm{K}} \neq 0$.
- For every $p \notin \mathrm{S}$, $\mathcal{H}_p$ acts on $\pi_p^{\mathrm{K}_p}$ by a character $\lambda_p(\pi)$ (Satake parameter).
- We say that π is S-CAP if there exists an Eisenstein series E on $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{K}$ such that $\lambda_p(\pi) = \lambda_p(\mathrm{E})$ for all $p \notin \mathrm{S}$.

Theorem A (R.B.P., Y. Liu, W. Zhang, X. Zhu)

Assume that π is not S-CAP. Then, there exists $\mu_\pi \in \mathcal{M}^{\mathrm{S}}(\mathrm{G})$ such that for every $f \in \mathcal{S}(\mathrm{G}(\mathbb{A}))_K$ we have :

1. $\mathrm{R}(\mu_\pi * f)$ *acts by zero on* $\mathrm{L}^2_{cusp}(\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}))^\perp$ *;*
2. $\pi(\mu_\pi * f) = \pi(f)$.

Remarks :

- If $\mathrm{G} = \mathrm{GL}_n$, every cuspidal representation is not S-CAP (Jacquet-Shalika) ;
- The proof is robust and allows for many variants (e.g. isolation of a cuspidal datum).
- This theorem has been applied in conjunction with Jacquet-Rallis trace formulas to prove the Gan-Gross-Prasad conjecture for unitary groups (see Pierre-Henri's talk).

Spectral description : Spherical Hecke Algebras

- For simplicity assume G split, fix a Borel $\mathrm{B} \subset \mathrm{G}$ and let $\mathrm{A} \twoheadleftarrow \mathrm{B}$ be the universal Cartan, $\mathrm{W} = \mathrm{W}(\mathrm{G}, \mathrm{A})$ the Weyl group and $\widehat{\mathrm{A}} = \mathrm{X}^*(\mathrm{A}) \otimes \mathbb{C}^\times$ the dual torus.
- For every p, we identify $\widehat{\mathrm{A}} \simeq \widehat{\mathrm{A}}_p$ with the gp of unramified chars of $\mathrm{A}(\mathbb{Q}_p)$ by $\chi \otimes p^s \mapsto |\chi|_p^s$.
- Every unramified irred repn π_p of $\mathrm{G}(\mathbb{Q}_p)$ (i.e. satisfying $\pi_p^{\mathrm{G}(\mathbb{Z}_p)} \neq 0$) has a **Satake parameter** $\lambda_p(\pi_p) \in \widehat{\mathrm{A}}_p / \mathrm{W}$ st π_p appears as a subquotient of $\mathrm{I}_{B(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(\lambda_p(\pi_p))$ (normalized induction).
- **Satake isomorphism** :

$$\mathcal{H}_p \simeq \mathbb{C}[\widehat{\mathrm{A}}_p]^W, \mu \mapsto \widehat{\mu}$$

  such that for every unramified irred repn π_p and $v \in \pi_p^{G(\mathbb{Z}_p)}$, $\pi_p(\mu)v = \widehat{\mu}(\lambda_p(\pi_p))v$.

Infinitesimal multipliers

- Similarly, for $\mu \in \mathcal{M}_\infty(\mathrm{G})$ and π_∞ : irred admissible repn of $\mathrm{G}(\mathbb{R})$ we have

$$\pi_\infty(\mu) = \widehat{\mu}(\pi_\infty)\,\mathrm{Id} \quad \text{(Schur)}$$

  and μ is characterized by $\pi_\infty \mapsto \widehat{\mu}(\pi_\infty)$.
- Harish-Chandra isomorphism :

$$\mathcal{Z}(\mathfrak{g}) \simeq \mathbb{C}[\mathrm{Lie}(\mathrm{A})^*_{\mathbb{C}}]^W \simeq \mathbb{C}[\mathrm{Lie}(\widehat{\mathrm{A}})]^W$$

  where $\mathcal{Z}(\mathfrak{g})$ denotes the center of the enveloping algebra of $\mathfrak{g}(\mathbb{C})$.
- Every π_∞ has an infinitesimal character $\lambda_\infty(\pi_\infty) \in \mathcal{Z}(\mathfrak{g})^\wedge \simeq \mathrm{Lie}(\widehat{\mathrm{A}})/\mathrm{W}$.
- We only look for multipliers $\mu \in \mathcal{M}_\infty(\mathrm{G})$ such that $\widehat{\mu}$ factorizes through $\pi_\infty \mapsto \lambda_\infty(\pi_\infty)$ (**infinitesimal multipliers**).

Tubular neighborhoods of the tempered spectrum

- $\widehat{\mathrm{G}(\mathbb{R})}^{\mathsf{temp}}$: set of all tempered irred. repns of $\mathrm{G}(\mathbb{R})$. Put

$$\mathsf{Inf}^{\mathsf{temp}} = \lambda_\infty\left(\widehat{\mathrm{G}(\mathbb{R})}^{\mathsf{temp}}\right) \subset \mathsf{Lie}(\widehat{\mathrm{A}})/\mathrm{W}.$$

- Harish-Chandra : π_∞ tempered iff $\exists\, \mathrm{B} \subset \mathrm{P} \subset \mathrm{G}$ with $\mathrm{P} \twoheadrightarrow \mathrm{M} \twoheadrightarrow \mathrm{A_M}$, σ discrete series of $\mathrm{M}(\mathbb{R})$ and $\lambda \in \sqrt{-1}\,\mathsf{Lie}(\mathrm{A_M})^*_{\mathbb{R}} \subset \mathsf{Lie}(\mathrm{A})^*_{\mathbb{C}} = \mathsf{Lie}(\widehat{\mathrm{A}})$ st

$$\pi_\infty \hookrightarrow \mathrm{I}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}(\sigma \otimes \lambda).$$

- "Tubular neighborhood" ($C > 0$) : $\pi_\infty \in \widehat{\mathrm{G}(\mathbb{R})}^{\mathsf{temp}}_{<C}$ iff $\exists\, \mathrm{B} \subset \mathrm{P} \subset \mathrm{G}$, σ d.s. of $\mathrm{M}(\mathbb{R})$ and $\lambda \in \mathsf{Lie}(\mathrm{A_M})^*_{\mathbb{C}}$ st $\|\Re(\lambda)\| < C$ and

$$\pi_\infty \hookrightarrow \mathrm{I}_{\mathrm{P}(\mathbb{R})}^{\mathrm{G}(\mathbb{R})}(\sigma \otimes \lambda).$$

- We set

$$\mathsf{Inf}^{\mathsf{temp}}_{<C} = \lambda_\infty\left(\widehat{\mathrm{G}(\mathbb{R})}^{\mathsf{temp}}_{<C}\right) \subset \mathsf{Lie}(\widehat{\mathrm{A}})/\mathrm{W}.$$

- Example : if $\mathrm{G} = \mathsf{SL}_2$, $\mathsf{Lie}(\widehat{\mathrm{A}})/\mathrm{W} = \mathbb{C}/\{\pm 1\}$ and we have

$$\mathsf{Inf}^{\mathsf{temp}} = i\mathbb{R}/\{\pm 1\} \cup \mathbb{Z}/\{\pm 1\},\ \ \mathsf{Inf}^{\mathsf{temp}}_{<C} = V_C/\{\pm 1\} \cup \mathbb{Z}/\{\pm 1\}$$

where V_C is the vertical band $|\Re(z)| < C$.

Construction of Archimedean multipliers

Theorem B (R.B.P., Y. Liu, W. Zhang, X. Zhu)

Let $\widehat{\nu}:\mathrm{Lie}(\widehat{A})\to\mathbb{C}$ *be holomorphic and such that*

- $\widehat{\nu}$ *is* W*-invariant ;*
- *For every* $C>0$, $\widehat{\nu}$ *is bounded by a polynomial on* $\mathrm{Inf}^{\mathrm{temp}}_{<C}$.

Then, there exists $\mu\in\mathcal{M}_\infty(G)$ *such that* $\widehat{\mu}(\pi_\infty)=\widehat{\nu}(\lambda_\infty(\pi_\infty))$ *for every irred adm repn* π_∞ *of* $G(\mathbb{R})$*. We denote the space of such multipliers by* $\mathcal{M}^{\mathrm{inf}}_\infty(G)$

- Arthur's multipliers : let $K_\infty\subset G(\mathbb{R})$ maxl compact subgroup then

$$C^\infty_c(G(\mathbb{R}))_{(K_\infty)}\curvearrowleft\mathcal{M}^A_\infty=\mathrm{PW}(\mathrm{Lie}(\widehat{A}))^W:=\mathcal{F}\,\mathcal{E}'(\mathrm{Lie}(A)_{\mathbb{R}})^W$$

  where (K_∞) means K_∞-finite (on both sides) and $\mathcal{F}$ is the Fourier transform. Not sufficient for our purpose (ess. b/c PW functions are bounded by an exponential).
- Delorme's multipliers :

$$\mathcal{S}(G(\mathbb{R}))_{(K_\infty)}\curvearrowleft\mathcal{M}^D_\infty=\mathcal{F}\,\mathcal{S}'_{\exp}(\mathrm{Lie}(A)_{\mathbb{R}})^W$$

  is $\pm$ what we want (fns of poly growth vertical strips but no growth condition in the real direction). However, need to show that the action of $\mathcal{M}^{\mathrm{inf}}_\infty$ extends by continuity to $\mathcal{S}(G(\mathbb{R}))\rightsquigarrow L^2$-argument (Plancherel formula)+translation by fin. diml repns.

On the proof of Theorem A

First recall the statement :

Theorem A

Let π be a cuspidal repn of $\mathrm{G}(\mathbb{A})$ that is not S-CAP. Then, there exists $\mu_\pi \in \mathcal{M}^{\mathrm{S}}(\mathrm{G})$ such that for every $f \in \mathcal{S}(\mathrm{G}(\mathbb{A}))_K$ we have :

1. $\mathrm{R}(\mu_\pi * f)$ *acts by zero on* $\mathrm{L}^2_{cusp}(\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A}))^\perp$;
2. $\pi(\mu_\pi * f) = \pi(f)$.

Set $\mathfrak{X}^S = \underbrace{\mathrm{Lie}(\widehat{\mathrm{A}})/\mathrm{W}}_{\text{inf. chars.}} \times \prod_{p \notin S} \underbrace{\widehat{\mathrm{A}}_p/\mathrm{W}}_{\text{Sat. param. at } p}$ and $\lambda^S(\pi) = (\lambda_\infty(\pi), (\lambda_p(\pi))_{p\notin \mathrm{S}}) \in \mathfrak{X}^S$.

- We define similarly $\lambda^{\mathrm{S}}(\mathrm{E})$ for an Eisenstein series E on $\mathrm{G}(\mathbb{Q})\backslash \mathrm{G}(\mathbb{A})/K$ and we set

$$\mathfrak{X}^S_{\text{Eis}} = \{\lambda^{\mathrm{S}}(\mathrm{E}) \mid \mathrm{E} \text{ Eis. series}\} \subset \mathfrak{X}^S.$$

- Note that $\mathcal{M}^{\mathrm{S,inf}}(\mathrm{G}) = \mathcal{M}^{\text{inf}}_\infty(\mathrm{G}) \bigotimes'_{p\notin \mathrm{S}} \mathcal{H}_p$ can be seen as a space of functions on $\mathfrak{X}^{\mathrm{S}}$ by $\mu \mapsto \widehat{\mu}$.
- We are looking for $\mu_\pi \in \mathcal{M}^{\mathrm{S,inf}}(\mathrm{G})$ such that $\widehat{\mu}_\pi$ vanishes on $\mathfrak{X}^S_{\text{Eis}}$ but not on $\lambda^S(\pi)$.

- Eisenstein series come in a countable number of families $\mathcal{F} = \{\mathrm{E}(\varphi_\lambda)\}_{\varphi,\lambda}$ where $\mathrm{B} \subset \mathrm{P} \subsetneq \mathrm{G}$, $\mathrm{P} \twoheadrightarrow \mathrm{M} \twoheadrightarrow \mathrm{A}_M$, $\varphi \in \sigma \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{M}(\mathbb{Q})\mathrm{N_P}(\mathbb{A})\backslash \mathrm{G}(\mathbb{A}))$ and $\lambda \in \mathrm{Lie}(\mathrm{A_M})^*_{\mathbb{C}} \subset \mathrm{Lie}(\widehat{\mathrm{A}})$.
- $\lambda^S(\mathcal{F})$ is then the image in $\mathfrak{X}^{\mathrm{S}}$ of a coset for

$$\left\{(\lambda, (p^\lambda)_{p\notin \mathrm{S}}) \mid \lambda \in \mathrm{Lie}(\mathrm{A_M})^*_{\mathbb{C}}\right\} \subset \mathrm{Lie}(\widehat{\mathrm{A}}) \times \prod_{p\notin \mathrm{S}} \widehat{\mathrm{A}}_p.$$

- Harish-Chandra finiteness : $\lambda_\infty(\pi) \notin \lambda_\infty(\mathcal{F})$ for all except a finite number of $\mathcal{F}$'s. Moreover, the $\lambda_\infty(\mathcal{F})$ are images of affine subspaces of $\mathrm{Lie}(\widehat{\mathrm{A}})$ that are quite "sparse" $\Rightarrow$ we can find $\mu_\infty \in \mathcal{M}^{\mathrm{inf}}_\infty(\mathrm{G})$ st $\widehat{\mu}_\infty$ vanishes on all of them except finitely many and $\widehat{\mu}_\infty(\lambda_\infty(\pi)) \neq 0$. (Here it is crucial to be able to choose $\widehat{\mu}_\infty$ of arbitrary growth in the real direction).
- For the remaining $\mathcal{F}$'s, we can separate $\lambda^{\mathrm{S}}(\pi)$ from $\lambda^{\mathrm{S}}(\mathcal{F})$ by using product of W-translates of functions of the form

$$(\lambda_\infty, (\lambda_p)_{p\notin \mathrm{S}}) \mapsto \chi(\frac{p^{\lambda_\infty}}{\lambda_p}) - c_{\chi,p}$$

where $\chi : \widehat{\mathrm{A}} \to \mathbb{C}^\times$ is a character and $c_{\chi,p} \in \mathbb{C}$.

Some open questions

Let $G/\mathbb{R}$ be a connected reductive group.

- Is $\mathcal{M}_\infty^{\mathsf{inf}}(G)$ the space of all infinitesimal multipliers ?
- Are there other elements in $\mathcal{M}_\infty(G)$? Besides infinitesimal multipliers we can also act by the center of $G(\mathbb{R})$ but e.g. when $G = PGL_2$ we don't have any multiplier separating the principal series

$$I_{B(\mathbb{R})}^{G(\mathbb{R})}(\lambda) \text{ and } I_{B(\mathbb{R})}^{G(\mathbb{R})}(\mathsf{sgn} \otimes \lambda)$$

  when $\lambda : \mathbb{R}_+^\times \to \mathbb{C}^\times$ is in generic position.
- What about a Paley-Wiener theorem for $\mathcal{S}(G(\mathbb{R}))$?

Thank you！