Scattering and a Plancherel formula of spherical varieties of real split reductive groups

Patrick Delorme

Happy birthday Bill

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Strategy due to Sakellaridis and Venkatesh in the p-adic case: introduce Bernstein maps (D., Knop, Kroetz, Schlichtkrull over $\mathbb{R}$) and then, with the help of an analog of their Discrete Series Conjecture, introduce scattering operators and prove their unitarity.

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One can define:

$$L^2(X_I)_{td} := \int_{i\mathfrak{a}_I^*}^{\oplus} L^2(X_I, \lambda)_{td} d\lambda. (measurability\ issue,\ see\ below) \tag{0.3}$$

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The conjecture is true in many cases for $I = S$ given by real analogs of cases given by Sakellaridis-Venkatesh in the p-adic case.

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Together with the conjecture and the description of $\mathbb{D}(X_I)$ above, this allows us to show that $L^2(X_I)_{td}$ **is the image of the spectral projection of** $\mathbb{D}(X_I)$ **attached to some part of its spectrum.**

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Bernstein morphisms are G-maps $i_I : L^2(X_I) \to L^2(X)$ given in the Plancherel isomorphisms (0.1) and (0.2) by the following Hilbert integrals of operators (D., Knop, Kroetz, Schlichtkrull):

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Together with the conjecture and the description of $\mathbb{D}(X_I)$ above, this allows us to show that $L^2(X_I)_{td}$ **is the image of the spectral projection of** $\mathbb{D}(X_I)$ **attached to some part of its spectrum.** For this, we use a property of spectral projections of type I groups, whose proof was given to me by Alain Connes. It says that, under some hypothesis: **the spectral projection for an Hilbert integral of representations is the Hilbert integral of the spectral projections**.This ensures measurabily of $L^2(X_I, \lambda)_{td}$ in λ.

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The Bernstein morphisms are abstract versions of wave packets of Eisenstein integrals of Harish-Chandra: abstract because the maps $j_{I,\pi} : \mathcal{M}_{I,\pi} \to \mathcal{M}_\pi$ are not explicit.

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Again, using spectral projections but for A^0_I and A^0_J, it is relatively easy to see that if $I \approx J$, $\exists$ operators (**scattering operators**) $S_{\mathfrak{w}} :: L^2(X_I)_{td} \to L^2(X_J)_{td}$, $\mathfrak{w}$ in the set $W_{I,J}$ of elements of W_X which conjugate $\mathfrak{a}_I$ to $\mathfrak{a}_J$, such that:

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$$i^*_{J,td} \circ i_{I,td} = \sum_{\mathfrak{w} \in W_{I,J}} S_{\mathfrak{w}}, \qquad (0.5)$$

and

$$S_{\mathfrak{w}} r(a_I) f = r(a_I^{\mathfrak{w}}) S_{\mathfrak{w}} f, \mathfrak{w} \in W_{I,J}, f \in L^2(X_I)_{td}, a_I \in A_I^0$$

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We will try, if time allows, to give some ingredient of the proof, after stating the main result, which follows from this unitarity, as in the work of Sakellaridis and Venkatesh.

9 Main Theorem

(i) If $I, J, K \subset S$, $I \approx J \approx K$:

$$i_{J,td} \circ S_{\mathfrak{w}} = i_{I,td}, \mathfrak{w} \in W_{I,J}.$$

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One ends up with a covering of X by a finite family of open sets of X, $U_{\mathfrak{i}} = U_{I,\mathfrak{i},\varepsilon_I}$, $I \subset S$, $\mathfrak{i} \in \mathfrak{I}$, ε_I measures the proximity to the boundary orbit Y_I.

11 Main tools: Main inequality, Approximate partition

In particular, U_i **is a subset of a translate of the open** P**-orbit in** X **which identifies with the same translate of the open** P**-orbit in each boundary degeneration of** X**.** In particular the constant terms of the restriction f_i of f to U_i might be viewed as functions on U_i.

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This leads to the unitarity of scattering operators.

The Main Theorem follows easily.