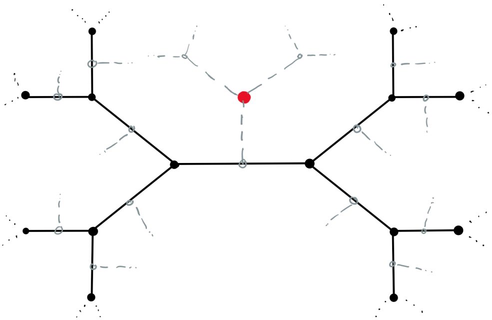
Representations of p-adic groups - with a twist

Jessica Fintzen

University of Cambridge and Duke University

November 2021



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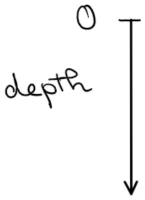

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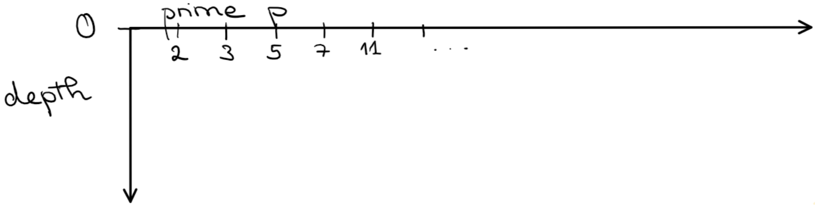

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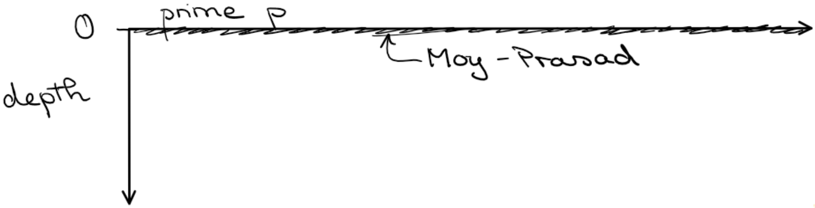

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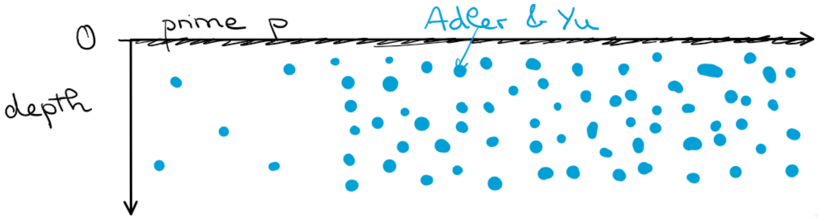

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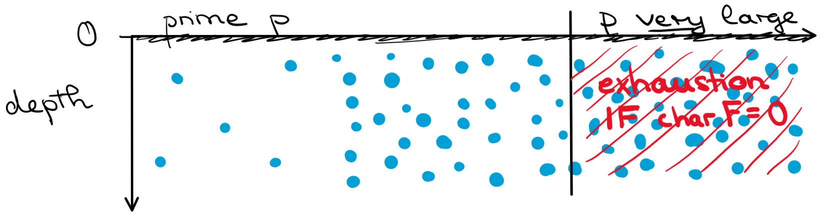

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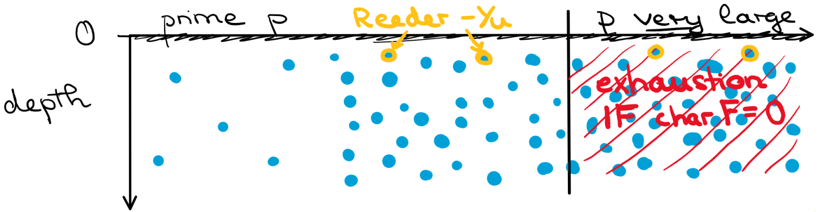

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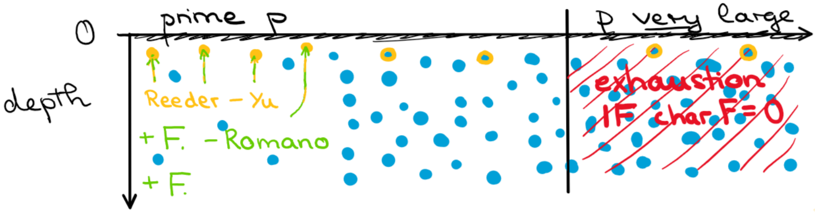

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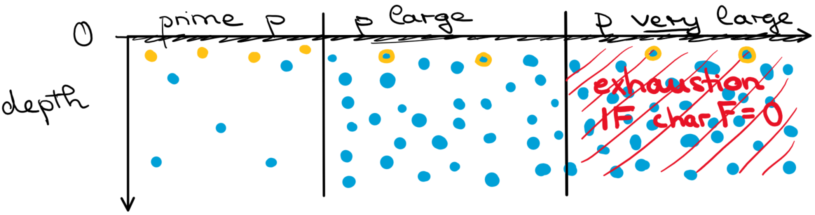

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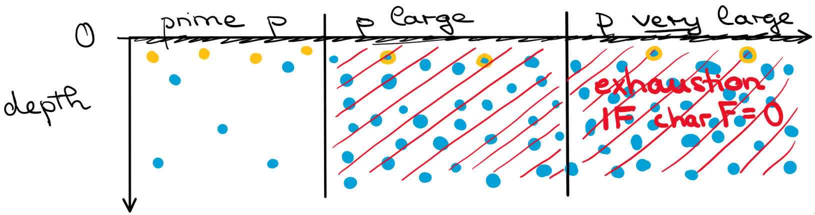

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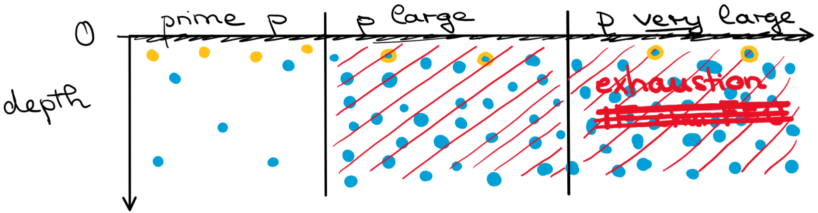

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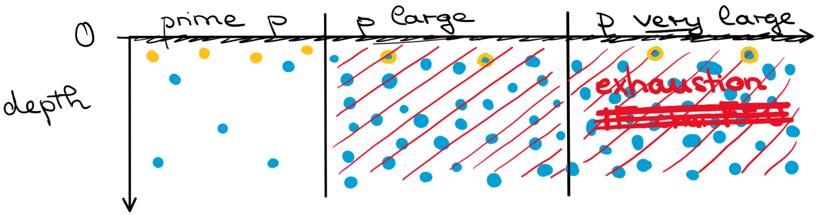
Suppose G splits over a tame extension of F and $p \nmid |W|$, then Yu's construction yields all supercuspidal representations.



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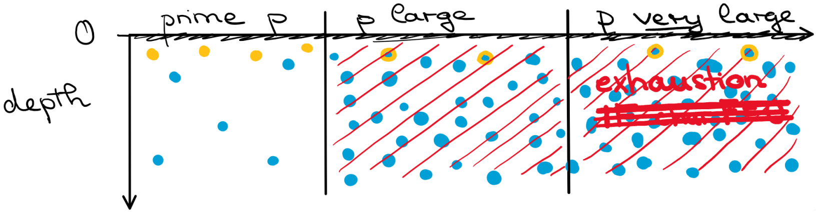
$\lvert W \rvert$	$(n+1)!$	$2^n \cdot n!$	$2^{n-1} \cdot n!$	$2^7 \cdot 3^4 \cdot 5$

type	E_7	E_8	F_4	G_2
$\lvert W \rvert$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	$2^7 \cdot 3^2$	$2^2 \cdot 3$



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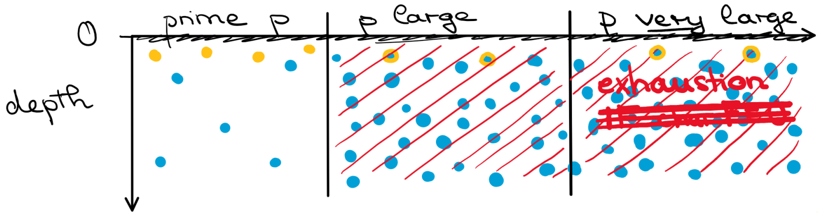

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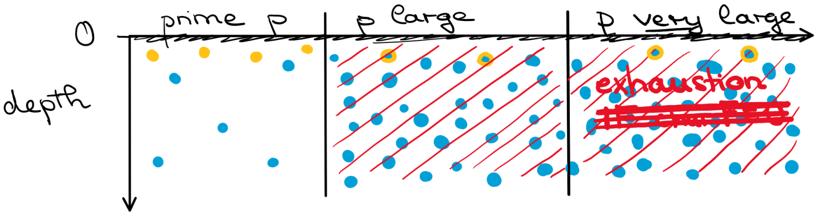

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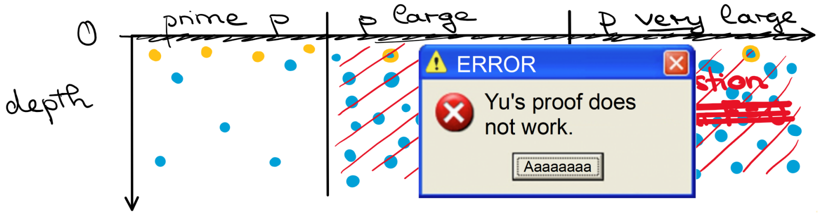

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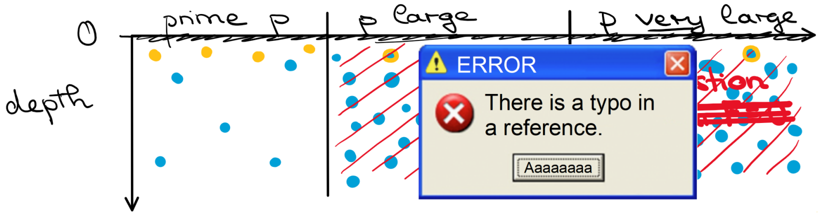

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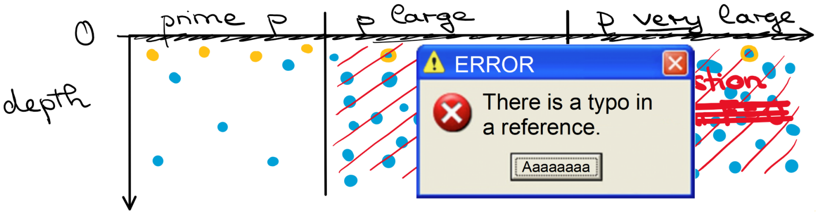

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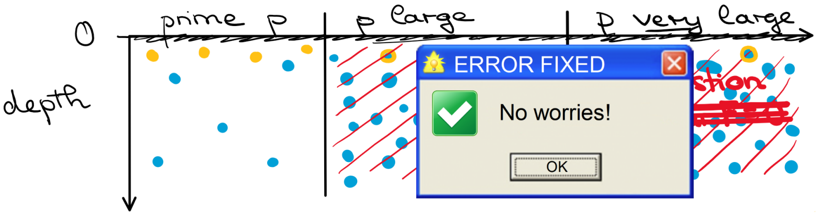

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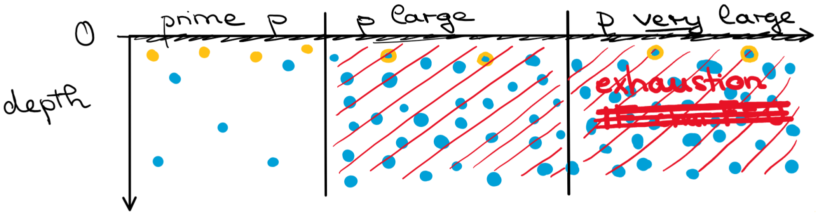

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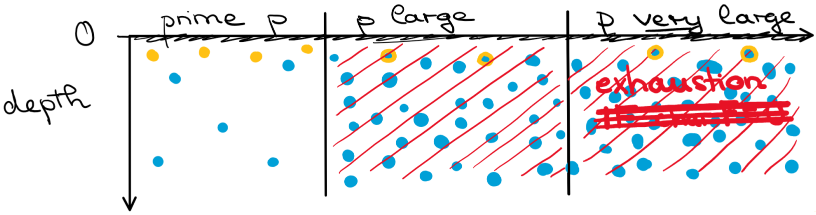
Theorem 5 (F.–Kaletha–Spice, June 2021) – vague version

We can twist Yu's construction such that Yu's Prop 14.1 and Thm 14.2 are satisfied for the twisted construction.



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2. Build a representation of G from the representation ρ_K (keyword: compact-induction).

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$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b$$

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$\bar{k} = \mathbb{C}$ or $\bar{\mathbb{F}}_\ell$, $G = \mathrm{SL}_2(F)$, $K = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \times \{\pm 1\}$

$\rho_K : K \to \mathrm{GL}_1(\bar{k}) = \bar{k}^*$, $\rho_K : \{\pm 1\} \to 1 \in \bar{k}^*$

$$\begin{aligned} & \quad\;\; G_{x,0.5} \qquad\qquad\qquad\qquad G_{x,0.5} \;/\; G_{x,0.5+} \\ \rho_K : & \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathcal{O} & 1+\mathfrak{p} \end{pmatrix} \Big/ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^2 \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \\ & \simeq \begin{pmatrix} 0 & \mathbb{F}_q \\ \mathbb{F}_q & 0 \end{pmatrix} \to \mathbb{F}_q \\ & \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \mapsto a + b \end{aligned}$$

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(Super)cuspidal representation:

$$\text{c-ind}_K^G \rho_K = \left\{ f : G \to \bar{k} \;\middle|\; \begin{array}{l} f(hg) = \rho_K(h) f(g) \ \forall g \in G, h \in K \\ f \text{ compactly supported} \end{array} \right\}$$

G-action: $g.f(\star) = f(\star \cdot g)$

Results continued continued

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There exists a counterexample to the key ingredient of Yu's proof (Yu's Prop 14.1 and Thm 14.2, which were based on a misprint).

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$$\epsilon_{\flat,0}^{G/M}(\gamma) = \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{asym}}/(\Gamma \times \{\pm 1\}) \\ \alpha_0 \in R(Z_M, G/M)_{\mathrm{sym,ram}} \\ 2 \nmid e(\alpha/\alpha_0)}} \mathrm{sgn}_{k_\alpha}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{sym,unram}}/\Gamma \\ \alpha_0 \in R(Z_M, G/M)_{\mathrm{sym,ram}} \\ 2 \nmid e(\alpha/\alpha_0)}} \mathrm{sgn}_{k_\alpha^1}(\alpha(\gamma))$$

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The end of the talk,
but only the beginning of the story ...