① Casselman's bday ← say happy b-day!

Beyond endoscopy and boundary terms w/
a view towards nonabelian trace formulae

§0 BE & PS

F = # field,

G = red F gp.

$r : {}^LG \to GL(V_r)$ a rep^n

Langlands Beyond Endoscopy proposal For $f \in C_c^\infty(G(\mathbb{A}_F))$

Study $\sum_\pi \operatorname{Res}_{s=1} L(s,\pi,r)\, \operatorname{tr} \pi(f)$ (*)

↑ cusp. rep^n of $G(\mathbb{A}_F)$

Why? Poles correspond to parameters

$\mathcal{L}_F \to \hat{H} \to {}^LG$

where $\hat{H}$ fixes a vector in V_r.

& $\hat{H}$ is "almost" a Langlands dual of an F-gp H.

try to compare w/ a suitable trace formulae over all
such H & establish cases of Langlands functoriality.

Poisson-summation conjecture

Braverman-Kazhdan, Ngô, L-Lafforgue, Schelleriedz (for spherical)

To r, associate M_r, a red. monoid w/

$M_r^\times = G$.

(2) There should exist $\mathcal{S}(M_r(\mathbb{A}_F)) = \otimes_v' \mathcal{S}(M_r(F_v)) \subseteq C^\infty(G(\mathbb{A}_F))$

+ a FT $\tilde{\mathcal{F}} : \mathcal{S}(M_r(\mathbb{A}_F)) \longrightarrow \mathcal{S}(M_r(\mathbb{A}_F))$

s.t. (1) $\sum_{\gamma \in G(F)} f(\gamma) + * = \sum_{\gamma \in G(F)} \mathcal{F}(f)(\gamma) + *$

(2) $\tilde{\mathcal{F}}$ is twisted equiv under $G(\mathbb{A}_F)^2$

(3) for $f = f_S b^S \in \mathcal{S}(M_r(\mathbb{A}_F))$

$\quad\quad\quad\quad$ ↳ basic f^{n}

$$\pi(f_S b^S) = \pi_S(f_S) L(\pi^S, r)\, \pi(\mathbb{1}_{G(\hat{\mathcal{O}}^S)})$$

If true, can follow Godement-Jacquet argument

to prove f^{n}al eq^{n} of $L(s, \pi, r)$

Varying r and applying converse theory

obtain much of Langlands functoriality.

Idea of Ngô: Integrate the PS conjecture to study BE.

For $f \in \mathcal{S}(M_r(\mathbb{A}_F))$, $f = f_S b^S$

$$\sum_{\gamma \in G(F)} f(g_1^{-1} \gamma g_2) = \sum_{\pi} L(\pi^S, r)\, K_{\pi(f_S \mathbb{1}_{G(\hat{\mathcal{O}}^S)})}(g_1, g_2)$$

Integrating over the diagonal + taking a residue,

obtain $*$.

In order for this to be helpful, need a geometric understanding

of the residue.

(3) Residues should corresp. to the "boundary terms" * in the PS formula.

Eg: $M_{st: GL_n \to GL_n} = M_{n\times n}$, boundary terms are

$$M_{n\times n} - GL_n$$

However, in the case where $M_{n\times n}$ is nonsmooth, boundary terms are not simply f^s on $M_r - G$.

(example later)

§1 The Rankin-Selberg monoid

Let $G(R) = \{(g_1, g_2) \in GL_2^2(R) \mid \det g_1 = \det g_2\}$

Take $r: {}^L G \to GL_4$, tensor product

$V_4(R) = M_r(R) = \{(X_1, X_2) \in gl_2^2(R) \mid \det X_1 = \det X_2\}$

This is (a) Rankin-Selberg monoid

Have a tower of quad spaces

~~ext of det~~ $\rightarrow$ ~~$V_4 \oplus G_a$~~ $\leftarrow$

$$\det \left\{ \begin{array}{ll} V_4 & = gl_2^{\oplus 2} \\ V_3 & = gl_2 \oplus \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} < gl_2 \\ V_2 & = gl_2 \\ V_1 & = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \end{array} \right.$$

(4) Using the tower we can give a PS formula with (some) boundary terms

X_i = zero locus of $\det|_{V_i}$

Have Schwartz space

$$S(X_i(\mathbb{A}_F)) = S(\underbrace{V_i(\mathbb{A}_F)}_{\text{Weil}} \oplus \underbrace{\mathbb{A}_F^2}_{\text{Standard}})_{SL_2(\mathbb{A}_F)}$$

$$\mathcal{F}_{X_i} : S(X_i(\mathbb{A}_F)) \longrightarrow S(X_i(\mathbb{A}_F))$$

~~Analogy~~

$$f_1 \otimes f_2 \longmapsto f_1 \otimes \mathcal{F}_2 f_2$$

standard SL_2^*-equiv. FT.

$$d_i : S(X_i(\mathbb{A}_F)) \longrightarrow S(X_{i-1}(\mathbb{A}_F))$$

$$f \longmapsto \mathcal{F}_2(\xi \mapsto f(\xi, 0, 0))$$

partial FT

For $z \in \mathbb{C}$

$$Z(\cdot, z) : S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \longrightarrow \mathbb{C}$$

$$f \longmapsto \int_{N(\mathbb{A}_F) \backslash SL_2(\mathbb{A}_F)} e^{H_B(g)(2-i-z)} r(g) f(0,0,1)\, dg.$$

$$I : S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)_{SL_2(\mathbb{A}_F)} \longrightarrow C^\infty(X_i^\circ(\mathbb{A}_F))$$

~~(integrate over SL_2~~

$$f_1 \otimes f_2 \longmapsto \left(\xi \mapsto \int_{N_2(\mathbb{A}_F) \backslash SL_2(\mathbb{A}_F)} \rho(g) f_1(\xi) f_2(g^{+}(0,1))\, dg\right)$$

⑤ **Theorem**[(G)] Let $f \in S(X_4(\mathbb{A}_F))$. Assume

$d_3 \circ d_4(f) = d_3 \circ d_4(\mathcal{F}_{X_4}(f)) = 0$. Then

$$Z_{\mathcal{E}_4}(f,-2) + Z_{\mathcal{E}_3}(d_4(f),-1) + \sum_{\xi \in X_4^\circ(F)} I(f)(\xi) + \sum_{\xi \in X_3^\circ(F)} I(d_4(f))(\xi)$$

$$= Z_{\mathcal{E}_4}(\mathcal{F}_{X_4}(f),-2) + Z(d_4(\mathcal{F}_{X_4}(f)),-1) + \sum_{\xi \in X_4^\circ(F)} I(\mathcal{F}_{X_4}(f))(\xi) + \sum_{\xi \in X_3^\circ(F)} I(d_4(\mathcal{F}_{X_4}(f)))\xi,$$

Rem: Works for all quadratic spaces of even dim. Obtained while working w/ Kazhdan in response to his questions.

Cor[(G)]: Under the same assumptions as above, + $R(f)$ has cusp. supp.,

$$\sum_{\pi} \mathrm{Res}_{s=1} L^S(s,\pi,r) K_{\pi(I_2(f))} I_{G(\hat{G}_F^S)}(g_1,g_2)$$

$$= c_F\left(Z_{\mathcal{E}_3}(d_3(\mathcal{F}_{X_4}(f)),-1) + \sum_{\xi \in X_3^\circ(F)} I(d_4(\mathcal{F}_{X_4}(f)))(\xi g_1 g_2)\right)$$

cusp. autom. rep^n of $G(\mathbb{A}_F) \subset GL_2(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$

⑥ §2 Nonabelian trace formulae Assume F/k Galois

$\langle \tau, \sigma \rangle = \mathrm{Gal}(F/k)$, e.g. any simple nonabelian gp.

The π on $G(\mathbb{A}_F) \subset GL_2^2(\mathbb{A}_F)$ that contribute above are

$\Pi \cong \pi_0 \otimes \pi_0^\vee$ where π_0 on $GL_2(\mathbb{A}_F)$.

$$\sum_{\pi_0} \mathrm{Res}_{s=1} L^S(s, \pi_0 \otimes \pi_0^\vee) K_\pi(g_1, g_2, g_3, g_4)$$

$$= * + c_F \sum_{\xi \in X_3^0(F)} I\left(d_4\left(\mathcal{F}_{X_0}\left((g_1, g_2)f(g_3, g_4)\right)\right)(\xi)\right)$$

Can integrate over $\{(g, \tau(g)), (h, \sigma(h))\}$

to obtain a geometric expression for

the set of cusp. autom repⁿ of $GL_2(\mathbb{A}_F)$ invariant

under $\langle \tau, \sigma \rangle$.

Will ~~Hope to~~ make the expression precise this spring.

w/ Taiyong Kang.

$\boxed{\text{Hope: can relate to a corresp. expression over } k \text{ \& study nonsolvable b. change.}}$

Softer, more accessible: Study # of repⁿs fixed by

$\mathrm{Gal}(F/k)$, already unknown.

(7) Rem: Even when we know the analytic cont of $L(s, \pi, \rho)$, this shows the BS conjecture gives <u>new</u> <u>information</u>. Could be transformative.

<u>Interesting, possibly accessible case</u>:

$$Up_{m,n} \backslash SL_{m+n} // Up^{op}_{m,n} \rightsquigarrow \text{RS monoid for } GL_m \times GL_n$$

(see). Wang for the geometry).

§3 <u>Sketch of the proof</u>

Have: $\tilde{\mathcal{F}}_2 : S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2) \longrightarrow S(V_i(\mathbb{A}_F) \oplus \mathbb{A}_F^2)$

FT in 2nd variable of $\mathbb{A}_F^2$.

Intertwines Weil on $V_i \oplus \mathbb{G}_a^2$ &

Weil $\otimes$ st$^\vee$ ~~on~~ $\mathbb{O}(V_i)$

Let

$$\Theta_f(g) = \sum_{\xi \in V_i \oplus \mathbb{G}_a^2(F)} \rho(g) f(\xi)$$

Have $\Theta_{\mathcal{F}_2^{-1}(f)} = \Theta_{\tilde{\mathcal{F}}_2^{-1}(\tilde{\mathcal{F}}_{X_4}(f))}$ by

PS.

(8) OTOH $\Theta_{\tilde{\mathcal{F}}_2^{-1}(f)}(g) = \sum_{\xi \in V_i(F) \oplus F^2} \rho \otimes st^\vee(g) . f(\xi)$

by PS.

So if $\Theta_{\tilde{\mathcal{F}}_2^{-1}(f)} \in L^1([SL_2])$

obtain $\int_{[SL_2]} \Theta_{\tilde{\mathcal{F}}_2^{-1}(f)}(g)\, dg = \int_{[SL_2]} \Theta_{\tilde{\mathcal{F}}_2^{-1}(\mathcal{F}_{X_4}(f))}(g)\, dg$

implies the theorem.

If not, replace $\Theta_{\tilde{\mathcal{F}}_2^{-1}(f)}$ w/ its Arthur truncation

Separating terms appropriately yields the formula w/

boundary terms. $\square$