Arthur's conjectures for symplectic and orthogonal similitude groups

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History

Société Mathématique de France
Astérisque 171-172 (1989), p.13-71.

UNIPOTENT AUTOMORPHIC REPRESENTATIONS: CONJECTURES

James Arthur

Foreword.

In these notes, we shall attempt to make sense of the notions of semisimple and unipc representations in the context of automorphic forms. Our goal is to formulate some conjectu both local and global, which were originally motivated by the trace formula. Some of these jectures were stated less generally in lectures [2] at the University of Maryland. The pre

(a) Asterisque, 1989

Unipotent Automorphic Representations: Global Motivation

JAMES ARTHUR

Contents

(b) Ann Arbor proceeding, 1990

Automorphic representations

- k number field, $\mathbb{A}$ adele ring of k.
- Γ_k absolute Galois group, W_k Weil group.
- G connected quasisplit reductive group over k.

Definition

Automorphic representations of $G(\mathbb{A})$ are irreducible constituents of the regular representation on $L^2(G(k)\backslash G(\mathbb{A}))$.

Fix central character $\xi : Z_G(k)\backslash Z_G(\mathbb{A}) \to \mathbb{S}^1$.

$$L^2(G(k)\backslash G(\mathbb{A}), \xi) = L^2_{disc}(G, \xi) \oplus L^2_{cont}(G, \xi)$$

Global Langlands Correspondence

$$\mathcal{A}(G) := \left\{ \substack{\text{automorphic} \\ \text{representations of } G(\mathbb{A})} \right\}_{/\sim} \longleftrightarrow \left\{ \substack{\text{L-parameters} \\ \phi: L_k \to {}^L G} \right\}_{/\widehat{G}-conj} =: \Phi(G)$$

L_k is the hypothetical global Langlands group satisfying

$$1 \longrightarrow C_k \longrightarrow L_k \longrightarrow W_k \longrightarrow 1$$

where C_k is compact. It is equipped with

$$L_{k_v} \to L_k$$

where

$$L_{k_v} = \begin{cases} W_{k_v} & k_v \text{ Archimedean} \\ W_{k_v} \times SU(2) & k_v \text{ non-Archimedean} \end{cases}$$

Global Langlands Correspondence

$$\begin{array}{ccc} L_k & \xrightarrow{\phi} & {}^L G \\ \uparrow & & \uparrow \\ L_{k_v} & \xrightarrow{\phi_v} & {}^L G_v \end{array}$$

$$\left\{ \begin{array}{c} \text{automorphic} \\ \text{representations of } G(\mathbb{A}) \end{array} \right\}_{/\sim} \longleftrightarrow \left\{ \begin{array}{c} \text{L-parameters} \\ \phi : L_k \to {}^L G \end{array} \right\}_{/\widehat{G}-conj}$$

Satake parameter $c_v = \{\phi_v(\mathrm{Fr}_v)\}$ at the unramified places

Local Langlands Correspondence

$$\mathrm{Irr}(G(k_v)) = \left\{ \begin{matrix} \text{irreducible admissible} \\ \text{representations of } G(k_v) \end{matrix} \right\}_{/\sim} \longrightarrow \left\{ \begin{matrix} \text{L-parameters} \\ \phi_v : L_{k_v} \to {}^L G_v \end{matrix} \right\}_{/\widehat{G}_v - conj} = \Phi(G_v)$$

$$\pi_v \mapsto \phi_{\pi_v}$$

When π_v is unramified, Satake parameter $c(\pi_v) = \{\phi_{\pi_v}(\mathrm{Fr_v})\}$.

This gives a partition of

$$\mathrm{Irr}(G(k_v)) = \bigsqcup_{\phi_v \in \Phi(G_v)} \Pi_{\phi_v}$$

where Π_{ϕ_v} are called local L-packets.

Local-global compatibility

Define

$$\Pi_\phi := \otimes'_v \Pi_{\phi_v}$$

Does Π_ϕ contain all automorphic representations corresponding to ϕ?

The answer is no.

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Vol. 33 (1979), part 1, pp. 315–322

A COUNTEREXAMPLE TO THE "GENERALIZED RAMANUJAN CONJECTURE" FOR (QUASI-) SPLIT GROUPS

R. HOWE AND I. I. PIATETSKI-SHAPIRO

1. Introduction. In **[Sat]**, Satake explains how the Ramanujan and Ramanujan-Peterson conjectures concerning the coefficients of cuspidal modular forms can be formulated group theoretically. Briefly, the interpretation is that the local constituents (see **[F]**) of the automorphic representation associated to a classical cusp form should be tempered in the sense of Harish-Chandra **[KZ]**. (See also **[GGP]**.)

To fix this, Arthur introduces the A-packets.

A-parameters

Conjecture

$$\mathcal{A}(G) \longleftrightarrow \Psi(G) := \left\{ \substack{\text{A-parameters} \\ \psi : L_k \times SL(2,\mathbb{C}) \to {}^L G \\ \psi(L_k) \text{ bounded}} \right\}_{/\widehat{G}-conj} \subseteq \Phi(G)$$

$$\psi \mapsto \phi_\psi(u) = \psi(u, \begin{pmatrix} |u|^{1/2} & 0 \\ 0 & |u|^{-1/2} \end{pmatrix})$$

$$\begin{array}{ccc} L_k \times SL(2,\mathbb{C}) & \xrightarrow{\psi} & {}^L G \\ \uparrow & & \uparrow \\ L_{k_v} \times SL(2,\mathbb{C}) & \xrightarrow{\psi_v} & {}^L G_v \end{array}$$

A-packets

Conjecture

For $\psi_v \in \Psi(G_v)$, one can associate a finite set Π_ψ of unitary irreducible admissible representation of $G(k_v)$ satisfying the following properties.

- $$\Pi_{\psi_v} \supseteq \Pi_{\phi_{\psi_v}}$$
- $$\Pi_\psi := \otimes'_v \Pi_{\psi_v}$$
  contains the automorphic representations corresponding to ψ.

We will assume that

$$\bigcap_v \mathrm{Ker}\{H^1(W_k, Z(\widehat{G})) \to H^1(W_{k_v}, Z(\widehat{G}_v))\} = 1.$$

Multiplicity formula

$\mathcal{S}_{\psi_v} = \pi_0(Z_{\widehat{G}_v}(\psi_v)/Z(\widehat{G}_v)^{\Gamma_{k_v}})$

$\Pi_{\psi_v} \to \mathrm{Rep}(\mathcal{S}_{\psi_v})$, $\pi_v \mapsto \epsilon_{\pi_v}$. If π_v is unramified, then ϵ_{π_v} is trivial.

$\iota_v : \mathcal{S}_\psi \to \mathcal{S}_{\psi_v}$

$\Pi_\psi \to \mathrm{Rep}(\mathcal{S}_\psi)$, $\pi \mapsto \epsilon_\pi = \otimes_v (\epsilon_{\pi_v} \circ \iota_v)$

$\Psi_2(G) := \{\psi \in \Psi(G) : |Z_{\widehat{G}}(\psi)/Z(\widehat{G})^{\Gamma_k}| < \infty\}$.

Conjecture

For $\pi \in \mathrm{Irr}_u(G(\mathbb{A}), \xi)$,

$$m_{disc}(\pi) = \sum_{\psi \in \Psi_2(G,\xi)} m_\psi(\pi)$$

$$m_\psi(\pi) = \begin{cases} \dim Hom_{\mathcal{S}_\psi}(\epsilon_\psi, \epsilon_\pi) & \text{if } \pi \in \Pi_\psi \\ 0 & \text{if } \pi \notin \Pi_\psi \end{cases}$$

$$\epsilon_\psi : \mathcal{S}_\psi \to \{\pm 1\}.$$

Invariant trace formula

$$I^G_{geo}(f) = I^G_{spec}(f), \quad f \in \mathcal{H}(G(\mathbb{A}))$$

The discrete part of $I^G_{spec}(f)$:

$$I^G_{disc}(f) = \sum_{\{M\}} |W(M)|^{-1} \sum_{w \in W(M)_{reg}} |\det(w-1)_{\mathfrak{a}^G_M}|^{-1} tr(M_P(w,\xi) I_P(\xi, f))$$

The summand for $M = G$ is the trace on the discrete spectrum.

Stablization

$$I^G_{disc}(f) = \sum_{H \in \mathcal{E}_{ell}(G)} \iota(G,H)\, S^H_{disc}(f^H),$$

where f^H is the Langlands-Shelstad transfer of f.

The upshot is that

$$S^G_{disc}(f) := I^G_{disc}(f) - \sum_{H \neq G \in \mathcal{E}_{ell}(G)} \iota(G,H)\, S^H_{disc}(f^H)$$

is **stable**.

Stable multiplicity formula

Conjecture

$$S^G_{disc}(f) = \sum_{\psi \in \Psi(G,\xi)} |\mathcal{S}_\psi|^{-1} \epsilon_\psi(s_\psi)\sigma(\bar{S}^0_\psi) f(\psi)$$

where s_ψ *is the image of* $\psi(1 \times -1)$ *in* $\mathcal{S}_\psi$ *and*

$$f(\psi) = \prod_v f_v(\psi_v)$$

and

$$f_v(\psi_v) := \sum_{\pi_v \in \Pi_{\psi_v}} \mathrm{tr}(\epsilon_{\pi_v}(s_{\psi_v})) f_{G_v}(\pi_v)$$

is stable.

Sign character ϵ_ψ

$$\psi : L_k \times SL(2,\mathbb{C}) \to {}^L G$$

$$\tau_\psi : Z_{\widehat{G}}(\psi) \times L_k \times SL(2,\mathbb{C}) \to {}^L G \xrightarrow{Ad} GL(\widehat{\mathfrak{g}})$$

$$\tau_\psi = \oplus_i \tau_i = \oplus_i (\lambda_i \otimes \mu_i \otimes \nu_i)$$

τ_i is said to be special if $\tau_i^\vee = \tau_i$ and $\epsilon(1/2, \mu_i) = -1$.

$$\epsilon_\psi(s) = \prod_{\tau_i \text{ special}} \det \lambda_i(s), \quad s \in Z_{\widehat{G}}(\psi).$$

Endoscopic character relations

$\Pi_{\psi_v} \to \mathrm{Rep}(\mathcal{S}_{\psi_v})$ is given by the endoscopic character relations.

For semisimple $s_v \in Z_{\widehat{G}_v}(\psi_v)$, $\widehat{H}_{s_v} := Z_{\widehat{G}_v}(s)^0$.

$$1 \to \widehat{H}_{s_v} \to {}^L H_{s_v} := \mathrm{im}(\psi_v) \cdot \widehat{H}_{s_v} \to W_{k_v} \to 1.$$

$$\begin{array}{ccc} L_{k_v} & \xrightarrow{\psi_v} & {}^L G_v \\ & {\scriptstyle \psi_{H_{s_v}}} \searrow \quad \nearrow {\scriptstyle \xi_v} & \\ & {}^L H_{s_v} & \end{array}$$

Conjecture

$$f_v^{H_{s_v}}(\psi_{H_{s_v}}) = \sum_{\pi_v \in \Pi_{\psi_v}} \mathrm{tr}(\epsilon_{\pi_v}(s_v s_{\psi_v})) f_{G_v}(\pi_v), \quad f_v \in \mathcal{H}(G(k_v))$$

where $f_v^{H_{s_v}}$ is the Langlands-Shelstad transfer of f_v.

Developments including inner forms

- $GL(n)$ (Mœglin-Waldspurger, Badulescu-Grbac)
- $Sp(n), SO(n)$ (Arthur, Taibi, Rui Chen-Jialiang Zou)
- $U(n)$ (Rogawski, Mok, Kaletha-Minguez-Shin-White)
- $GSp(4)$ (Gee-Taibi)
- G_2 (Gan-Gurevich-Jiang, Gan)
- $Mp(2n)$ (Gan-Ichino)
- Global rigid inner forms (Kaletha)

Symplectic and even orthogonal similitude groups

$G = Sp(2n)$ (resp. $SO(2n)$), $\widetilde{G} = GSp(2n)$ (resp. $GSO(2n)$)

$$1 \longrightarrow G \longrightarrow \widetilde{G} \xrightarrow{\lambda} \mathbb{G}_m \longrightarrow 1$$

On the dual side

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow GSpin(N, \mathbb{C}) \xrightarrow{\mathbf{p}} SO(N, \mathbb{C}) \longrightarrow 1$$

$$\bar{\Psi}_2(\widetilde{G}, \widetilde{\zeta}) \to \bar{\Psi}_2(G, \zeta)$$

whose fiber admits a transitive action by

$$Y := \mathrm{Hom}_{cont}(\mathbb{A}^\times / k^\times, \mathbb{Z}/2\mathbb{Z}).$$

Multiplicity formula

Conjecture

For any $\psi \in \bar{\Psi}_2(G,\zeta)$*, there exists a global A-packet* $\bar{\Pi}_{\tilde{\psi}}$ *with central character* $\widetilde{\zeta}$ *unique up to twist by* Y *such that*

$$\bar{\Pi}_{\tilde{\psi}} = \otimes'_v \bar{\Pi}_{\tilde{\psi}_v}$$

and

$$L^2_{disc}(\widetilde{G}(F)\backslash\widetilde{G}(\mathbb{A}),\widetilde{\zeta}) = \widehat{\bigoplus}_{\psi\in\Psi_2(G,\zeta)} \widehat{\bigoplus}_{\omega\in Y/Y(\tilde{\psi})} \widehat{\bigoplus}_{\tilde{\pi}\in\bar{\Pi}_{\tilde{\psi}}\otimes\omega} m_{\tilde{\psi}}(\tilde{\pi})\,\tilde{\pi}$$

as $\bar{\mathcal{H}}(G)$*-module.*

Theorem (X. 2021)

The conjecture holds for the tempered part.

Local A-packets

Let F be a p-adic field, $\tilde{\psi} \in \Psi(\widetilde{G}, \tilde{\zeta})$ and $\psi = \mathbf{p} \circ \tilde{\psi}$.

$$\mathcal{S}_\psi \cong (\mathbb{Z}/2\mathbb{Z})^r, \quad \mathcal{S}_{\tilde{\psi}} \hookrightarrow \mathcal{S}_\psi$$

$$\bar{\Pi}_\psi \to \mathrm{Irr}(\mathcal{S}_\psi) \quad \text{(Mœglin)}$$

Let

$$\tilde{\bar{\Pi}}_{\psi,\tilde{\zeta}} := \{\tilde{\pi} \in \mathrm{I}\bar{\mathrm{r}}\mathrm{r}(\widetilde{G}(F), \tilde{\zeta}) : \tilde{\pi}|_{G(F)} \subseteq \bar{\Pi}_\psi\}.$$

Define

$$\tilde{\bar{\Pi}}_{\psi,\tilde{\zeta}} \to \mathrm{Irr}(\mathcal{S}_{\tilde{\psi}}), \quad \tilde{\pi} \mapsto \epsilon_{\tilde{\pi}} := \epsilon_\pi|_{\mathcal{S}_{\tilde{\psi}}}$$

where π is any irreducible representation in $\tilde{\pi}|_{G(F)}$.

Local A-packets

Theorem (X. 2021)

There exists a subset $\bar{\Pi}_{\tilde{\psi}}$ *of* $\tilde{\bar{\Pi}}_{\psi,\tilde{\zeta}}$ *such that*

1. $$\bigoplus_{\tilde{\pi}\in\bar{\Pi}_{\tilde{\psi}}} \tilde{\pi}|_{G(F)} = \bigoplus_{\pi\in\bar{\Pi}_{\psi}} \pi,$$

2. $$\tilde{f}(\tilde{\psi}) := \sum_{[\tilde{\pi}]\in\bar{\Pi}_{\tilde{\psi}}} \epsilon_{\tilde{\pi}}(s_{\tilde{\psi}})\tilde{f}_{\widetilde{G}}(\tilde{\pi}), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}(F))$$

   is stable.

3. *Suppose* $\tilde{\psi}$ *factors through* $\tilde{\psi}' \in \Psi(\widetilde{G}'_s)$ *for semisimple* $s \in Z_{\widehat{\widetilde{G}}}(\tilde{\psi})$, *then*
   $$\tilde{f}'(\tilde{\psi}') = \sum_{\tilde{\pi}\in\bar{\Pi}_{\tilde{\psi}}} \epsilon_{\tilde{\pi}}(ss_{\tilde{\psi}})\tilde{f}_{\widetilde{G}}(\tilde{\pi}), \quad \tilde{f} \in \bar{\mathcal{H}}(\widetilde{G}(F))$$

   where $\tilde{f}'$ *is the Langlands-Shelstad transfer of* $\tilde{f}$.

Infinitesimal character

The infinitesimal character of $\pi \in \mathrm{Irr}(G(F))$ is defined to be that of ϕ_π, namely the $\widehat{G}$-conjugacy class of

$$\lambda : W_F \to {}^L G, \quad w \mapsto \phi_\pi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right).$$

Previously we have associated $\tilde{\phi} \in \bar{\Phi}(\widetilde{G})$ with an L-packet $\bar{\Pi}_{\tilde{\phi}}$ unique up to twists by quadratic characters. We can make choices so that the parabolic induction preserves the infinitesimal character.

Proposition

For any Levi subgroup $\widetilde{M}$ of $\widetilde{G}$ and infinitesimal character $\lambda_{\widetilde{M}}$,

$$\mathrm{Ind}_{\widetilde{P}(F)}^{\widetilde{G}(F)} \bar{\mathrm{Irr}}(\widetilde{M}(F))_{\lambda_{\widetilde{M}}} \subseteq \bar{\mathrm{Irr}}(\widetilde{G}(F))_{\tilde{\lambda}}$$

for $\tilde{\lambda} = \iota_{\widetilde{M}} \circ \lambda_{\widetilde{M}}$, where $\iota_{\widetilde{M}} : {}^L\widetilde{M} \to {}^L\widetilde{G}$.

This kind of statement is due to Haines. In the case of classical groups, this has been proved by Moussaoui.

Construction

$$\mathrm{Irr}(\widetilde{G}(F))_{\tilde{\lambda}} \longleftrightarrow \mathrm{Irr}(G(F))_{\lambda}$$

- If the fiber over $\mathrm{Irr}(G(F))_{\lambda}$ are singletons, we define

$$\bar{\Pi}_{\tilde{\psi}} = \{\tilde{\pi} \in \bar{\mathrm{Irr}}(\widetilde{G}(F))_{\tilde{\lambda}} : \tilde{\pi}|_{G(F)} \subseteq \bar{\Pi}_{\psi}\}.$$

- In general, we extend Mœglin's contruction of $\bar{\Pi}_{\psi}$ to $\bar{\Pi}_{\tilde{\psi}}$, which reduces to the tempered case.

Stable multiplicity one

Question

Is the space of stable distributions supported on $\bar{\Pi}_\psi$ spanned by $f(\psi)$?

Theorem (X. 2021)

Suppose

$$\psi = \bigoplus_i \rho \otimes Sym^{a_i - 1} \otimes Sym^{b_i - 1} \in \Psi(G)$$

such that ρ is self-dual of orthogonal type, $a_i + b_i$ is even and $a_i \geqslant b_i$. Let $A_i = (a_i + b_i)/2 - 1$ and $B_i = (a_i - b_i)/2$. If

$$A_{i+1} \geqslant A_i \text{ and } B_{i+1} \geqslant B_i \text{ for all } i \ ,$$

then the space of stable distributions supported on $\bar{\Pi}_\psi$ (resp. $\tilde{\bar{\Pi}}_\psi$) is spanned by $f(\psi)$ (resp. $\tilde{f}(\tilde{\psi} \otimes \omega)$ for all quadratic characters ω).