A formula for the kernel of the ρ-Fourier transform

Bill Casselman's 80th birthday conference, Nov. 2021

Ngô Bảo Châu, University of Chicago

report on work in progress with Z. Luo

Langlands' automorphic L-function

G reductive group over global field k

$\rho : {}^LG \longrightarrow GL(V_\rho)$ finite dim'l rep.

π automorphic rep. of G, $\pi = \bigotimes_{v \in |k|} \pi_v$

$$L(s, \pi, \rho) = \prod_v L(s, \pi_v, \rho)$$

- defined as a Euler product ranging at unramified places

Conj
- admit meromorphic continuation
- admit functional equation after inserting "right" local factors at the remaining places.

- follows from Langlands' functoriality conj. and Godement-Jacquet's theory of principal L-function.
- Converse theorem.

Godement-Jacquet-Tate theory

$G = GL_1$, $\rho : GL_1 \xrightarrow{id} GL_1$ Tate's thesis

$G = GL_n$, $\rho : GL_n \xrightarrow{id} GL_n$ Godement-Jacquet

Ingredients

- $GL_n \subset M_n$ space of $n \times n$ matrices
- $\mathcal{S}(M_n(F)) \ni \mathbb{1}_{M_n(\mathcal{O})}$ $F = k_v, \mathcal{O} = \mathcal{O}_v$
- Fourier transform on $\mathcal{S}(M_n(F))$
- Poisson summation formula

$$\Big\Downarrow \text{ Mellin transform}$$

Principal L-functions

Braverman-Kazhdan proposal

G red group / k, $\rho : {}^LG \to GL(V_\rho)$

- Define a monoid $M^\rho \supset G$ as an open subset of invertible elements.
- Define a space of functions $\mathcal{S}^\rho(G(F))$ of smooth function on $G(k_v)$ with relatively compact support in $M(F)$ $\boxed{F = k_v}$ + extra conditions.
- Define a basic function $\beta_v^\rho \in \mathcal{S}^\rho(G(F))$ defining unramified L-factor.
  (IC-function on the arc space $\mathcal{L}M^\rho$ — Bouthier-Ngo-Sakellaridis)
- Define the ρ-Fourier transform
- Poisson summation formula

ρ- Fourier transform: expectation

$\varphi \longmapsto K^\rho * \varphi$ $\quad K^\rho$ kernel

- K^ρ stably invariant distribution
- K^ρ essentially relatively compact support in $M^\rho(k_v)$
- K^ρ given by integration of some algebraic functions.
- K^ρ acts on matrix coefficient by scalar multiplication by Γ_ρ-factor.

  $\Rightarrow$ compatible with parabolic descent
- involutive and unitary
- satisfy Poisson summation formula

Torus case (not necessarily split)

$G = T$ torus , $\rho : {}^L T \longrightarrow GL(V_\rho)$
$${}^L T = \hat{T} \rtimes \Gamma$$

$\Gamma \circlearrowleft$ multiset of weights of $\hat{T}$ in V_ρ

$\Rightarrow$ induced torus $\subset$ induced affine space

$$\begin{tikzcd}
D^\rho \arrow[r, hook] \arrow[d, "\rho^\vee"'] & A^\rho \arrow[r, "tr"'] \arrow[d] & \mathbb{A}^1 \\
T \arrow[r, hook] & M^\rho &
\end{tikzcd}$$

Monoid $M^\rho = A^\rho /\!/ D^\rho_1$

toric variety $\quad D^\rho_1 = \ker(D^\rho \to T)$

$$J^\rho_T(t) = \int_{\rho^{\vee -1}(t)} \psi(tr(x))\, d^\times x$$

for appropriate choice of invariant measure $d^\times x$ on $\rho^{\vee -1}(t)$.

- J^{ρ}_{t} is defined by an "algebraic integration", (with exponential ψ)

- It acts on irreducible representation by the correct γ-factor.

- It has essentially relatively compact support in M^{ρ}

- It satisfies the Poisson summation formula.

<u>Conclusion</u>: $K^{\rho}_{T} = J^{\rho}_{T}$ in the torus case.

J- function on the Steinberg base

G reductive split group, $\rho : {}^L G \to GL(V_\rho)$

T_0 maximal split torus

W acts on the multiset of weights of T in V_ρ $\Rightarrow$ W acts on D^ρ

$$\begin{tikzcd} D^\rho \arrow[r] \arrow[d] & D /\!/ W \arrow[d] \\ T_0 \arrow[r] & T_0 /\!/ W \end{tikzcd}$$

$\longrightarrow$ function $J^\rho_{T/\!/W}$ on $T/\!/W(F)$ which interpolates function J^ρ_T for all maximal tori T of G.

$\longrightarrow$ stably invariant function on G

<u>Question</u> : $K^\rho = J^\rho_{T/\!/W}$?

Compatibility with parabolic descent

$K^\rho \neq J^\rho_{T/\!/W}$ for the latter is not compatible with parabolic descent

L. Lafforgue : how to correct this defect when $G = GL_2$

$g \in GL_2$, $\quad a_1(g) = \mathrm{tr}(g)$
$\quad a_2(g) = \det(g)$

. $J^\rho_{T/\!/W}$ can be seen as function of variable a_1, a_2

. Fourier transform on variable a_1

$$\hat{J}^\rho_{T/\!/W}(\alpha_1, a_2) = \int_F J^\rho_{T/W}(a_1, a_2)\, \psi(-\alpha_1 a_1)\, da_1$$

$$K^\rho(a_1, a_2) = \int_F |\alpha_1|\, \hat{J}^\rho_{T/\!/W}(\alpha_1, a_2)\, \psi(\alpha_1 a_1)\, d\alpha_1$$

(Lafforgue's formula involves $|a_2|^{\dim(V_\rho)}$ ...)

"Pseudo-differential" operator

(joint work with Zhilin Luo)

We expect that

$$J^{\rho}_{T/\!/W} \rightsquigarrow K^{\rho}$$

invariant operator independent of ρ

pseudo-differential

We write down a formula for this invariant pseudo-differential operator for $G = GL_n$.

$g \in GL_n$, $a_i(g) = tr(\wedge^i g)$

Fourier transform on variable

$a_1, \dots, a_n \longrightarrow$ dual variable $\alpha_1, \dots, \alpha_n$

The Fourier transform on Steinberg base was considered in Frenkel-Langlands-Ngô.

$$\hat{J}^{\rho}_{T/\!/W}(\alpha_1, \dots, \alpha_n) = \int_{F^n} J^{\rho}_{T/\!/W}(a_1, \dots, a_n)$$

$$\psi(-\langle a, \alpha\rangle)\, da$$

$$K^{\rho}_{GL_n}(a_1, \dots, a_n) =$$

$$\int_{F^n} \hat{J}^{\rho}_{T/\!/W}(\alpha)\, |D_n(a, \alpha)|\, \psi(\langle a, \alpha\rangle)\, d\alpha$$

$D_n(a, \alpha)$ is a polynomial of variables $a_1, \dots, a_n, \alpha_1, \dots, \alpha_n$ defined as follows $\Leftrightarrow$ W_n-invariant polynomial $D_n(t, \alpha)$ where $t = \mathrm{diag}(t_1, \dots, t_n)$

$$D_n(t, \alpha) = \prod_{I \in \mathcal{I}_{n-2}(n)} \left(\sum_{i=1}^{n-1} \alpha_i \, \mathrm{tr}(\wedge^{i-1} t_I) \right)$$

$$I = \{ i_1 < \dots < i_{n-2} \} \subset \{1, \dots, n\}$$

$$t_I = \mathrm{diag}(t_{i_1}, \dots, t_{i_{n-2}})$$

Provisional result

$$(K^\rho * \varphi)_N = J^\rho * \varphi_N$$

for φ smooth compactly supported in the big cell.

Descent formula $\quad GL_n \longrightarrow GL_{n-1} \times G_m$

$$x = \begin{pmatrix} 1_{n-1} & 0 \\ v^- & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} & v \\ 0 & t_n \end{pmatrix}$$

$v, v^- \in F^{n-1}$

$$\boxed{\begin{aligned} &\int_{F^{n-1}} \int_{F^{n-1}} K^\rho_n(x)\, \varphi(v^-)\, dv^- dv \\ &= |t_n|^{-(n-1)} K^\rho_{n-1}(x_{n-1}, t_n)\, \varphi(0) \end{aligned}}$$

Key calculation

$$K^{\rho}_{n}(a) = \int_{F^n} \hat{J}_{T/\!/W}(\alpha) \left| D_n(a,\alpha) \right| \psi(\langle \alpha, a \rangle)\, d\alpha.$$

Calculate

$$\int_{F^{n-1}} \int_{F^{n-1}} \psi\Big(\sum_{i=1}^{n} \alpha_i \; a_i(x)\Big)\, \psi(v^-)\, dv^- \, du$$

Calculate $x \in GL_n$, $a_i^{[n]}(x) = \mathrm{tr}\, \Lambda^i x$

$x_{n-1} \in GL_{n-1}$, $a_i^{[n-1]}(x_{n-1}) = \mathrm{tr}\, \Lambda^i x_{n-1}$

$$x = \begin{pmatrix} 1_{n-1} & 0 \\ v^- & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} & v \\ 0 & t_n \end{pmatrix}$$

$$a_i^{[n]}(x) = a_i^{[n]}\begin{pmatrix} x_{n-1} & 0 \\ 0 & t_n \end{pmatrix}$$

$$+ \underbrace{c_{i-1}(x_{n-1}, v, v^*)}$$

bilinear function on v, v^*

homogeneous of degree $i-1$ on x_{n-1}

$$\sum_{i=1}^{n} \alpha_i^{[n]} a_i^{[n]} \begin{pmatrix} x_{n-1} & 0 \\ 0 & t_n \end{pmatrix}$$

$$= \alpha_1^{[n]} t_n + \sum_{i=1}^{n-1} \Big(\underbrace{\alpha_i^{[n]} + t_n \alpha_{i+1}^{[n]}}_{\alpha_i^{[n-1]}} \Big) a_i^{n-1}(x_{n-1})$$

$$\det\Big(\sum_{i=1}^{n} \alpha_i \, \underline{\tilde{C}_{i-1}(x_{n-1}, u, v^*)} \Big)$$

Matrix with entries function on gl_{n-1}

$$= \frac{D_n(a^{[n]}, \alpha^{[n]})}{D_{n-1}(a^{[n-1]}, \alpha^{[n-1]})}$$

---

what's next :

- complete descent thm
- involutivity, unitarity
- Poisson summation formula
- other groups.

Happy 80th birthday, Bill.

Looking forward to your 90th birthday