Asymptotic bounds for the homology of arithmetic lattices (joint work with M. Frączyk and S. Hurtado)

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Torsion homology

In all the talk : G is a semisimple Lie group ; $X = G/K$ its symmetric space.

Theorem

There exists a constant C depending only on G such that for any torsion-free arithmetic lattice Γ in G and $0 \leq i \leq d = \dim(X)$

$$\log |H_i(\Gamma, \mathbb{Z})_{\mathrm{tors}}| \leq C \operatorname{vol}(\Gamma \backslash G).$$

- Torsion-freeness of Γ is likely not necessary but removing the assumption would require additional work.
- Arithmeticity of Γ is necessary only for $G = \mathrm{SO}(3,1)$ since Bader–Gelander–Sauer proved a similar result for negatively curved manifolds in dimensions > 3.
- The statement is sharp "in general" ; maybe not for all G.

Betti numbers

Theorem

For $0 \leq i \leq \dim(X)$ *there exists a function* f_i *depending only on* G, i *such that* $f_i(v) = o(v)$ *and for any torsion-free* congruence *arithmetic lattice* Γ

$$\dim H_i(\Gamma, \mathbb{C}) \leq f_i(\mathrm{vol}\,\Gamma \backslash G), \ i \neq d/2$$

$$\left| \dim H_{d/2}(\Gamma, \mathbb{C}) - \beta_{d/2}^{(2)}(X)\,\mathrm{vol}(\Gamma \backslash G) \right| \leq f_{d/2}(\mathrm{vol}\,\Gamma \backslash G).$$

- The constants $\beta_{d/2}^{(2)}(X)$ have explicit formulas.
- The "congruencity" hypothesis is necessary for all groups $\mathrm{SO}(n,1), n \geq 3$ and for $\mathrm{SU}(2,1)$ at least (likely for all $\mathrm{SU}(n,1)$ as well).
- The f_i can be made more explicit with additional hypotheses (maybe in this generality as well).

The Bergeron–Venkatesh conjecture

Conjecture

For $0 \leq i \leq \dim(X)$ *there exists a function* h_i *depending only on* G, i *such that* $\lim_{v \to +\infty} h_i(v)/v = 0$ *and for any torsion-free* congruence *arithmetic lattice* Γ *in* G *and* $i \neq (d-1)/2$ *(*$d = \dim(X)$*)*

$$\log |H_i(\Gamma, \mathbb{Z})_{\mathrm{tors}}| \leq h_i(\mathrm{vol}\, \Gamma \backslash G)$$

and

$$\left| \log |H_{(d-1)/2}(\Gamma, \mathbb{Z})_{\mathrm{tors}}| - t^{(2)}(X)\, \mathrm{vol}(\Gamma \backslash G) \right| \leq h_{(d-1)/2}(\mathrm{vol}\, \Gamma \backslash G).$$

Brief interlude on arithmetic and congruence subgroups

Roughly speaking an arithmetic subgroup Γ of G is a finite-index subgroup of some $\mathrm{G}(\mathbb{Z}_k)$ where k is a number field with ring of integers $\mathbb{Z}_k$ and G a $\mathbb{Z}_k$-group scheme with $\mathrm{G}(\mathbb{R}) \cong G$ (...)

A congruence subgroup comes from pulling back a subgroup of a finite group $\mathrm{G}(R)$ obtained via a morphism from to a finite ring R.

Arithmetic groups are not always congruence, for example if there is a surjective morphism $\mathrm{G}(\mathbb{Z}_k) \to S_n$ for large n the kernel cannot be congruence. Finding out for which G (or G) this is the case is the congruence subgroup problem which still has some outstanding open cases.

Some constructions of non-congruence subgroups violate the "sharp" asymptotic estimates on Betti numbers, and those on torsion homology in the Bergeron–Venkatesh conjecture.

General bounds for torsion homology

The degree of a simplicial complex is the graph degree of its 1-skeleton (i.e. every vertex is a neighbour to at most (degree) vertices).

Lemma

Let $D, i \in \mathbb{N}$. There exists a constant $C_{D,i}$ such that if M is a space which is homotopy equivalent to a simplicial complex with v vertices and degree at most D then

$$\log |H_i(M, \mathbb{Z})_{\mathrm{tors}}| \leq C_{D,i} \cdot v.$$

The proof rests on the following lemma.

Lemma

If A is an integer matrix with m columns all of which are of Euclidean norm $\leq C$ then

$$|\mathrm{coker}(A)_{\mathrm{tors}}| \leq C^m.$$

Homotopy type of arithmetic manifolds

Theorem (Gelander's conjecture)

There exists constants C_X, D_X such that any arithmetic X-manifold M is homotopy equivalent to a simplicial complex of degree at most D_X and with at most $C_X \cdot \mathrm{vol}(M)$ vertices.

This implies immediately that $\dim H_i(\Gamma, \mathbb{C}) \ll_X \mathrm{vol}(\Gamma \backslash X)$ for torsion-free lattices (which was already known by an old theorem of Gromov–Ballmann–Schoen), and the result on torsion via the previous slide.

Triangulations of Riemannian manifolds

Lemma

There exists a constant C_d such that any d-dimensional Riemannian manifold M with sectional curvatures in $[-1, 1]$ is homotopy equivalent to a simplicial complex of degree at most C_d and with at most

$$C_d \int_M \max(1, \mathrm{inj}_x(M))^{-d} dx$$

vertices.

Corollary

The Gelander conjecture holds for X if we have $C, \varepsilon > 0$ such that

$$\mathrm{vol}(M_{\leq \varepsilon}) \leq C \, \mathrm{vol}(M) \cdot \mathrm{inj}(M)^d$$

for all compact arithmetic X-manifolds M.

Thin part of arithmetic locally symmetric spaces

Given a lattice $\Gamma \leq G$ we denote by k_Γ its trace field—this is the smallest totally real number field k such that there is a k-group G such that $\mathrm{G}(\mathbb{R}) \cong G$ and $\Gamma \leq \mathrm{G}(k)$.

Theorem

For any X and any $R > 0$ there are $C_{X,R}, \eta > 0$ such that for all arithmetic X-manifolds $M = \Gamma \backslash X$ we have

$$\mathrm{vol}(M_{\leq R}) \leq C_{X,R} e^{-\eta [k_\Gamma : \mathbb{Q}]} \mathrm{vol}(M).$$

Lemma (Dobrowolski)

For all compact arithmetic X-manifolds $M = \Gamma \backslash X$ we have

$$\mathrm{inj}(M) \geq \frac{c_X}{(\log [k_\Gamma : \mathbb{Q}])^3}.$$

$(\log [k_\Gamma : \mathbb{Q}])^{3d} e^{-c[k_\Gamma : \mathbb{Q}]} = o(1)$ so Gelander's conjecture is proven.

Benjamini–Schramm covergence and Betti numbers

The estimates on the thin part also imply the finer statement on Betti numbers via the machinery of Benjamini–Schramm convergence and its applications to limit multiplicities, when $[k_\Gamma : \mathbb{Q}] \to +\infty$.

For lattices with trace field of constant degree the proof uses the same machinery but the input is given by very different methods.

Arithmetic refinement of the Margulis lemma

The following theorem is a consequence of a result of Breuillard.

Theorem ("Arithmetic Margulis Lemma")

There exists ε_G such that for any uniform arithmetic lattice Γ in G with trace field k_Γ and any $x \in X$ the subgroup

$$\langle \gamma \in \Gamma,\, d(x, \gamma x) \leq \varepsilon_G \cdot [k_\Gamma : \mathbb{Q}] \rangle$$

is virtually abelian.

Applications of the arithmetic Margulis lemma

Given $\gamma \in \Gamma$ and $R \leq \varepsilon_G[k_\Gamma : \mathbb{Q}]$ we have the "model Margulis tube"

$$T_{\gamma,R} = Z_\Gamma(\gamma)\backslash\{x \in X : d(x, \gamma x) \leq R\}.$$

These tubes can a priori intersect and self-intersect in the manifold $M = \Gamma\backslash X$ but the AML implies the following lower bound for $\mathrm{vol}(M_{\leq R})$ (the upper bound is trivial).

Lemma

There exists m depending only on G such that for $R \leq \varepsilon_G[k_\Gamma : \mathbb{Q}]$

$$\sum_{[\gamma]\subset\Gamma} \mathrm{vol}(T_{\gamma,R}) \geq \mathrm{vol}(M_{\leq R}) \geq [k_\Gamma : \mathbb{Q}]^{-m} \sum_{[\gamma]\subset\Gamma} \mathrm{vol}(T_{\gamma,R})$$

(sum is over all nontrivial semisimple conjugacy classes in Γ).

Volume of tubes and orbital integrals

The orbital integral attached to a semiisimple element $\gamma \in G$ and a compactly supported continuous function f is

$$O(\gamma, f) = \int_{Z_G(\gamma)\backslash G} f(x^{-1}\gamma x)dx$$

If $f = 1_{B_G(R)}$ then $f(x^{-1}\gamma x^) = 1$ iff $d(x, \gamma x) \leq R$ so that

Lemma

$\mathrm{vol}(T_{\gamma,R}) = O(\gamma, 1_{B_G(R)}) \cdot \mathrm{vol}(Z_\Gamma(\gamma)\backslash Z_G(\gamma))$.

Main estimate for orbital integrals

Theorem

There exists $C_1, C_2, \delta > 0$ all depending only on G such that for any semisimple $\gamma \in G$ and $R, S > 0$ we have

$$O(\gamma, 1_{B_G(C_1 R + S)}) \geq C_2 e^{\delta S} O(\gamma, 1_{B_G(R)}).$$

This follows from direct computation on the group G, using different decompositions of measure according to whether γ generated a bounded or unbounded subgroups.

Corollary

$\mathrm{vol}(T_{\gamma, C_1 R + S}) \geq C_2 e^{\delta S} \mathrm{vol}(T_{\gamma, R})$

Conclusion

$$\mathrm{vol}(M) \geq \mathrm{vol}(M_{\leq \varepsilon_G [k_\Gamma : \mathbb{Q}]}) \geq [k_\Gamma : \mathbb{Q}]^{-m} \sum_{[\gamma]} \mathrm{vol}(T_{\gamma, \varepsilon_G [k_\Gamma : \mathbb{Q}]})$$

(by corollary to AML)

$$\gg_X [k_\Gamma : \mathbb{Q}]^{-m} \sum_{[\gamma]} e^{\varepsilon_G \delta [k_\Gamma : \mathbb{Q}] - C_1 R} \mathrm{vol}(T_{\gamma, R})$$

(by corollary to estimates of orbital integrals)

$$\gg_{R,X} e^{\frac{\varepsilon_G \delta}{2} [k_\Gamma : \mathbb{Q}]} \mathrm{vol}(M_{\leq R}).$$