A simulative approach to the construction of new Ricci solitons

Timothy Buttsworth

Singularity Models of the Ricci Flow

A *gradient shrinking Ricci soliton* is a triple (M, g, u), where M is a smooth manifold, g is a Riemannian metric, and $u : M \to \mathbb{R}$ is a smooth function so that

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- the round three-sphere (u is constant);
- the Gaussian (flat Euclidean space with $u(x) = \frac{|x|^2}{2}$);
- the shrinking cylinder ($M = \mathbb{S}^2 \times \mathbb{R}$ with standard metric, $u(x, y) = \frac{y^2}{2}$).

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There does not appear to be any known non-round Ricci solitons on $\mathbb{S}^4$.

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If one simply wanted to find *new* solitons (rather than a classification), one could use more efficient stochastic techniques to detect new solutions, and then use Leray-Schauder degree theory to confirm existence.

$SO(3) \times SO(2)$-invariant solitons on $\mathbb{S}^4$

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$$dt^2 + f_1^2\mathbb{S}^1 + f_2^2\mathbb{S}^2, \tag{1}$$

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The IVP is controlled by one real parameter at one end, and two real parameters at the other end.

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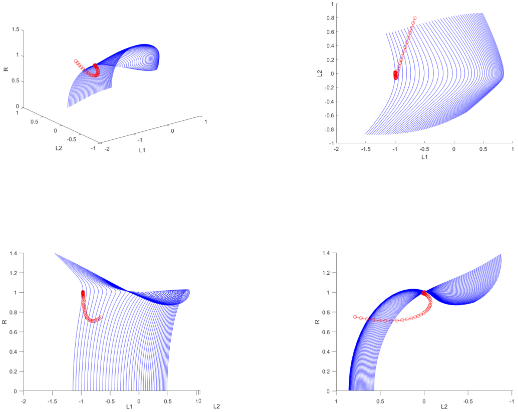
As a shooting problem:



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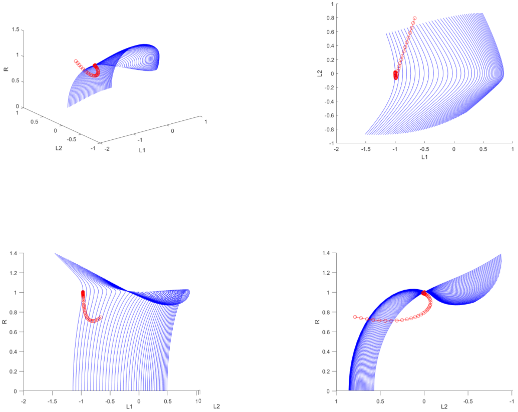

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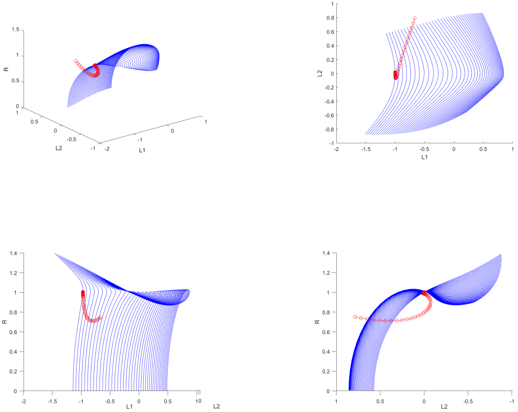

The obvious intersection is the round Einstein metric. There appears to be a sequence of 'almost' intersections with unbounded Riemann curvature.

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Theorem

There exists a $\mathcal{C} > 0$ so that any $SO(3) \times SO(2)$-invariant soliton on $\mathbb{S}^4$ has Riemann curvature bounded pointwise by $\mathcal{C}$.

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Conjecture

The only $SO(3) \times SO(2)$-invariant soliton on $\mathbb{S}^4$ is the round metric, up to diffemorphism.

Could be upgraded to 'Theorem' with enough computational power.

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Conjecture

Possibly a new ancient solution?

Near this sequence of 'almost Ricci solitons'.