Grothendieck-Verdier duality in categories of VOA modules

with examples

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Outline

1) Tensor structures and VOAs

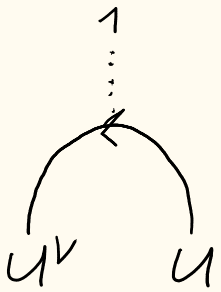
2) Grothendieck-Verdier categories

3) Free bosons as an example

Recall rigidity in tensor categories:

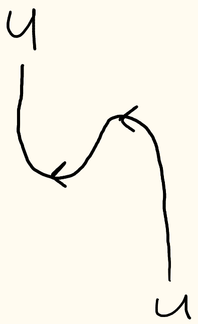
For an object U find a dual object $U^\vee$

and morphisms

evaluation 

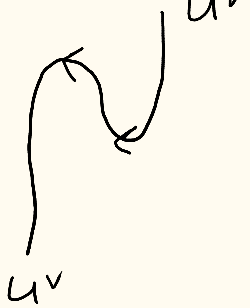
, coevaluation 

satisfying the "straightening relations"



$=$

and



$=$

This is a property: $(U^\vee, ev, coev)$ unique, if they exist.

Results on rigidity from VOA theory are rare:

- rational theories yield MTCs — Hung '08
- $W_{1,p}$ triplet is rigid — Tsuchiya, SW '13
- Flatness of simple currents — Creutzig, Kanade, Linshaw, Ridout '16
- Virasoro, affine (super) algs, $\beta\gamma$ admit choices of rigid categories — Creutzig, McRae, Kanade, Yang, Jiang, Orosz Hunziker, Ridout, Allen, SW '20

VOA tensor structures in a nutshell [Huang-Lepowsky-Zhang '13]

<u>Def</u> For M_1, M_2, M_3 V-modules, a <u>(logarithmic) intertwining operator</u> of type $\binom{M_3}{M_1, M_2}$ is a linear map

VOA

$$\mathcal{Y}: M_1 \otimes_{\mathbb{C}} M_2 \to M_3\{z\}[\log z]$$
$$m_1 \otimes m_2 \mapsto \mathcal{Y}(m_1, z) m_2$$

such that

1) lower truncation: lower bounds on z-powers.
2) L_{-1} derivation: $\frac{d}{dz}\mathcal{Y}(-, z) = \mathcal{Y}(L_{-1}-, z)$
3) Jacobi identity:

"morally" $Y_{M_3}(v, w)\,\mathcal{Y}(m_1, z) m_2 \simeq \mathcal{Y}(Y_{M_1}(v, w-z) m_1, z) m_2$

$$\simeq \mathcal{Y}(m_1, z)\, Y_{M_2}(v, w) m_2$$

Remark

The action Y_M of V on M is a int op of type $\binom{M}{V, M}$.

Normalised: $Y_M(\mathbb{1}, z) = \mathrm{id}_M$

Int ops are V-bilinear maps!

Def Tensor product (aka fusion)

Let M_1, M_2 be V-modules. Their tensor product is a pair

$$\left(M_1 \boxtimes M_2,\ \mathcal{U}_{M_1,M_2}\right)$$

int op of type $\binom{M_1 \boxtimes M_2}{M_1,\ M_2}$

s.t.

$$\begin{tikzcd}
M_1 \otimes_{\mathbb{C}} M_2 \arrow[r, "\mathcal{U}_{M_1,M_2}"] \arrow[dr, "\mathcal{Y}_X"'] & M_1 \boxtimes M_2\{z\}[\log z] \arrow[d, dashed, "\exists!"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & X\{z\}[\log z]
\end{tikzcd}$$

Construction of $M_1 \boxtimes M_2$ highly non-trivial.

HLZ method: Construct dual (characterised by inward arrows)

See review by [Kanade, Ridout '18]

Categorical properties that follow (with assumptions)

Thm (HLZ)

Let $\mathcal{C}$ be a category of V-modules with $V \in \mathcal{C}$ and

1) $\forall M_1, M_2 \in \mathcal{C}$, $(M_1 \boxtimes M_2, \mathcal{Y}_{M_1,M_2})$ exists.

2) $\forall M_1, M_2, M_3 \in \mathcal{C}$ exist a family of isomorphisms

$$\mathcal{A}^{x_1,x_2}_{M_1,M_2,M_3} : (M_1 \boxtimes M_2) \boxtimes M_3 \to M_1 \boxtimes (M_2 \boxtimes M_3)$$

s.t

$$\mathcal{A}^{x_1,x_2}_{M_1,M_2,M_3}\left(\mathcal{Y}_{M_1,M_2\boxtimes M_3}(-,x_1)\,\mathcal{Y}_{M_2,M_3}(-,x_2)-\right)$$

Product $\to$ Iterate

$$= \mathcal{Y}_{M_1\boxtimes M_2,M_3}\left(\mathcal{Y}_{M_1,M_2}(-,x_1-x_2)-,x_2\right)$$

Then,

1) $-\boxtimes-$ defines a functor via: for $f: M_1 \to N_1$, $g: M_2 \to N_2$,
$f \boxtimes g$ is the unique morphism satisfying

$$\mathcal{U}_{N_1,N_2} \circ (f \otimes_{\mathbb{C}} g) = (f \boxtimes g) \circ \mathcal{U}_{M_1,M_2}.$$

2) V is tensor unit. Unit morphisms, e_M, r_M characterised by

$$e_M(\mathcal{U}_{V,M}(v,z)m) = Y_M(v,z)m$$

$$r_M(\mathcal{U}_{M,V}(m,z)v) = e^{zL_{-1}} Y_M(v,-z)m$$

3) Associator

$$A_{M_1,M_2,M_3} = \lim_{x_2 \to 1} \lim_{x_1 - x_2 \to 1} A^{x_1,x_2}_{M_1,M_2,M_3}$$

4) Braiding $c_{M_1,M_2} : M_1 \boxtimes M_2 \longrightarrow M_2 \boxtimes M_1$ characterised via

$$c_{M_1,M_2}\left(\mathcal{Y}_{M_1,M_2}(m_1, z)\, m_2 \right) = e^{zL_{-1}}\, \mathcal{Y}_{M_2,M_1}(m_2, e^{i\pi} z)\, m_1$$

$$\Rightarrow \binom{M_3}{M_1,\ M_2} \simeq \binom{M_3}{M_2,\ M_1}$$

5) Twist $\theta_{M_1} = e^{2\pi i L_0}\big|_{M_1}$: tensor equivalence [id functor] $\longrightarrow$ [double braiding]

i.e satisfies balancing property

$$\theta_{M_1 \boxtimes M_2} = c_{M_2,M_1} \circ c_{M_1,M_2} \circ (\theta_{M_1} \boxtimes \theta_{M_2})$$

Where does rigidity come from?

So $\binom{M}{V, M} \simeq \binom{V'}{M', M}$ gives natural candidate for evaluation if $V \simeq V'$.

Let (M, Y_M) V-module.

Contagredient module $(M', Y_{M'})$, where

$$M' = \bigoplus_n \mathrm{Hom}(M_n, \mathbb{C}),$$

choose grading s.t. M_n finite dim

For any $\varphi \in M'$, $m \in M$, $v \in V$

$$\langle Y_{M'}(v)\varphi, m\rangle = \langle \varphi, Y_M^{opp}(v) m\rangle$$

$$Y_M^{opp}(v) = Y_M\left(e^{zL_1}(-z^{-2})^{L_0} v, z^{-1}\right)$$

Prop (HLZ)

$M'' \simeq M$

Generaliz to int ops by replacing $Y_M \in \binom{M}{V, M}$ by an int op of type $\binom{M_3}{M_1, M_2} \Longrightarrow \binom{M_2'}{M_1, M_3'} \simeq \binom{M_3}{M_1, M_2}$

Grothendieck-Verdier categories

<u>Def</u> Let $\mathcal{C}$ be a tensor category.

$K \in \mathcal{C}$ is <u>dualising</u> if

1) the functor $\mathrm{Hom}(- \otimes Y, K)$ is representable, ie.

$\forall Y \in \mathcal{C}, \exists\, DY \in \mathcal{C}$ s.t.

$$\mathrm{Hom}(X \otimes Y, K) \simeq \mathrm{Hom}(X, DY) \quad \forall X \in \mathcal{C}.$$

2) The contravariant functor $D : Y \mapsto DY$ characterised by 1 is an antiequivalence.

D is the <u>duality functor</u> with resp to K.

$(\mathcal{C}, K)$ is called a <u>Grothendieck-Verdier (GV)</u> category.

Prop (Boyarchenko Drinfeld '11)

Let $(\mathcal{C}, K)$ be a GV-cat.

1) Let $u \in \mathcal{C}$ be invertible, then
$Du \simeq K \otimes u^{-1}$ & $D^{-1}u \simeq u^{-1} \otimes K$ are dualising

$D\mathbb{1} \simeq K$

$D\mathbb{1} \simeq \mathbb{1}$ not assumed

2) The functor $u \mapsto K \otimes u^{-1}$ & $u \mapsto u^{-1} \otimes K$
are antiequivalences $\{\text{invertibles}\} \longleftrightarrow \{\text{dualisers}\}$

dualisers are a torsor over invertibles

3) Internal homs exist. For $X, Y \in \mathcal{C}$

$$\forall X, Y \in \mathcal{C} \qquad \underline{\mathrm{Hom}}(X, Y) \simeq D(X \otimes D^{-1} Y)$$

4) If $\mathcal{C}$ is braided with twist θ, then
$\theta^D = D^{-1} \theta_{DX}$ is also a twist.
$(\mathcal{C}, K, \theta)$ is $\underline{\text{ribbon GV}}$ if $\theta^D = \theta$.

D defines involution on twists.

Connection to VOAs

Thm (Allen-Lentner-Schweigert-Sw '21)

Let V be a VOA, $\mathcal{C}$ a category of VOA modules s.t. HLZ theory applies and which is closed under $'$ (contragredients). Then V' is a dualising object and $(\mathcal{C}, V', e^{2\pi i L_0})$ is ribbon GV.

<u>Proof</u> We know from above $\forall X, Y, Z \in \mathcal{C}$

$$\mathrm{Hom}(X \boxtimes Y, Z) \sim \binom{Z}{X\ \ Y} \simeq \binom{Y'}{X\ \ Z'} \sim \mathrm{Hom}(X \boxtimes Z', Y').$$

Set $Z = V'$, then

$$\mathrm{Hom}(X \boxtimes Y, V') \simeq \mathrm{Hom}(X \boxtimes V'', Y') \underset{V \cong V''}{\simeq} \mathrm{Hom}(X \boxtimes V, Y') \underset{r_X}{\simeq} \mathrm{Hom}(X, Y').$$

Thus V' is dualising with $D = {}'$. $\Theta = \Theta^D$ follows from $L_0^{opp} = L_0$:

$$D^{-1}(e^{2\pi i L_0})_{DM} = \left(e^{2\pi i L_0^{opp}}\big|_{M'}\right)' = \left(e^{2\pi i L_0}\big|_{M'}\right)' = e^{2\pi i L_0}\big|_M$$

∎

Example: Free bosons (following Dong-Lepowsky's book)

Data: - $\mathfrak{h}$ fin dim $\mathbb{R}$-vector space

- $\langle \, , \rangle$ sym non-deg form on $\mathfrak{h}$ — not assuming pos def

- discreet subgroup $\Lambda \subset \mathfrak{h}$ s.t. $\langle \, , \rangle|_\Lambda$ even integral — not assuming $\langle \, , \rangle|_\Lambda$ non-deg or $\Lambda \neq 0$ or $\mathrm{rk}\,\Lambda = \dim \mathfrak{h}$

- distinguished element $\xi \in \Lambda^*$ (Feigin-Fuchs boson), where $\Lambda^* = \{ \mu \in \mathfrak{h} \mid \langle \mu, \Lambda \rangle \subset \mathbb{Z} \}$.

Also need: - (set theoretic) section $s: \Lambda^*/\Lambda \to \Lambda^*$ (normalised $s(\Lambda) = 0$)

and aux map $k: \Lambda^*/\Lambda \times \Lambda^*/\Lambda \to \Lambda$
$(\mu, \nu) \mapsto s(\mu+\nu) - s(\mu) - s(\nu)$ } failure of s to be a hom.

- Normalised 2-cocycle $\varepsilon: \Lambda \times \Lambda \to \mathbb{C}^\times$ with Skew $(-1)^{\langle \alpha, \beta \rangle}$, $\alpha, \beta \in \Lambda$.

All choices cohomologous

Construct lattice VOA with GV cat of modules in the following steps.

1) Affinise $\mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}_{\mathbb{C}}$ as triv Lie alg to get rank $= \dim \mathfrak{h}$ Heisenberg Lie alg $\Big\}$ $\alpha, \beta \in \mathfrak{h}_{\mathbb{C}}$ $[\alpha_m, \beta_n] = m \langle \alpha, \beta \rangle \delta_{m+n,0} \mathbb{1}$.

2) Weight 0 Fock space, $\mathcal{F}_0$ becomes vertex algebra via

$$Y(\alpha_{-1}|0\rangle, z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1}, \quad \alpha \in \mathfrak{h}_{\mathbb{C}}.$$

Any basis of $\mathfrak{h}$ is a set of strong generators.

For any $\gamma \in \mathfrak{h}_\gamma$

$$\omega_\gamma = \frac{1}{2} \sum_i \alpha^i_{-1} \alpha^{i*}_{-1} |0\rangle + \gamma_{-2}|0\rangle$$

is a conformal vector with central charge

$$c_\gamma = \dim \mathfrak{h} - 12 \langle \gamma, \gamma \rangle$$

$\forall \beta \in \mathfrak{h}_{\mathbb{C}}$ $\beta_0 |_{\mathcal{F}_\alpha} = \langle \beta, \alpha \rangle \mathrm{id}_{\mathcal{F}_\alpha}$

Every Fock space $\mathcal{F}_\alpha$, $\alpha \in \mathfrak{h}_{\mathbb{C}}$ is a simple $\mathcal{F}_0$-module of conformal weight $h_\alpha = \frac{1}{2} \langle \alpha, \alpha - 2\gamma \rangle$

3) $\dim \begin{pmatrix} & \mathcal{F}_\rho & \\ \mathcal{F}_\mu, & & \mathcal{F}_\nu \end{pmatrix} = \begin{cases} 1 & \text{if } \rho = \mu + \nu \\ 0 & \text{else} \end{cases}$

Intertwining operators are propotional to "vertex operators" (normally ordered exponentials)

$$I_{\mu,\nu} \quad F_\mu \otimes_{\mathbb{C}} F_\nu \longrightarrow \mathcal{F}_{\mu+\nu}[[z, z^{-1}]]\, z^{\langle \mu,\nu \rangle}$$
$$|\mu\rangle \otimes |\nu\rangle \longmapsto z^{\langle \mu,\nu\rangle}\big(|\mu+\nu\rangle + \mathcal{O}(z)\big)$$

4) Lattice VOA $\quad V(\xi, \Lambda) = \bigoplus_{\alpha \in \Lambda} \mathcal{F}_\alpha \qquad$ ($\xi \in \Lambda^*$ ensures integral grading)

Fieldmap $\quad Y|_{\mathcal{F}_\mu \otimes_{\mathbb{C}} \mathcal{F}_\nu} = \varepsilon(\mu,\nu)\, I_{\mu,\nu}, \quad \mu,\nu \in \Lambda$

Conformal vector ω_ξ (central charge and categorical data only depend on $\xi + \Lambda$)

5) Lattice Fock spaces $\quad \mathbb{F}_\rho = \bigoplus_{\alpha \in \Lambda} \mathcal{F}_{s(\rho)+\alpha}, \quad \rho \in \Lambda^*/\Lambda$. Simple over $V(\xi, \Lambda)$.

Thm (Mostly Dong-Lepowsky + Tuite-Zuevsky, Li-Wang, Creutzig-Kanade-Linshaw-Ridout, Allen-Lentner-Schweigert-Sw, probably more)

Let $VM(\mathfrak{h}, \langle -,- \rangle, \Lambda, \xi)$ be the category of generalised Λ^*-graded $V(\xi, \Lambda)$ modules which are finitely generated, $\mathfrak{h}$-semisimple & real, $\hat{\mathfrak{h}}_+$ locally nilpotent.

Then HLZ theory applies and $VM(\mathfrak{h}, \langle -,- \rangle, \Lambda, \xi)$ is ribbon GV with

1) Fusion $\mathcal{F}_\mu \boxtimes \mathcal{F}_\nu = \mathcal{F}_{\mu+\nu}$, $\mu, \nu \in \Lambda^*/\Lambda$

$$U_{\mu,\nu}\Big|_{S(\mu)+\alpha_1,\, S(\nu)+\alpha_2} = (-1)^{\langle S(\mu), \alpha_2 \rangle} \varepsilon(\alpha_1, \alpha_2)\, \varepsilon(\alpha_1+\alpha_2, k(\mu,\nu)) \cdot I_{S(\mu)+\alpha_1,\, S(\nu)+\alpha_2}$$

2) Braiding $c_{\mu,\nu} = e^{i\pi \langle S(\mu), S(\nu) \rangle}$

3) Associativity $A_{\mu,\nu,\rho} = (-1)^{\langle S(\mu), k(\nu,\rho) \rangle} \dfrac{\varepsilon(k(\mu,\nu), k(\mu+\nu,\rho))}{\varepsilon(k(\nu,\rho), k(\mu, \nu+\rho))}$

4) Duals $\mathbb{F}_\rho' = \mathbb{F}_{2\xi-\rho}$, $\rho \in \Lambda^*/\Lambda$.

Dualising object $\mathbb{F}_{2\xi}$

5) Twist $\theta_{\mathbb{F}_\rho} = e^{i\pi \langle s(\rho), s(\rho) - 2\xi \rangle}$

The end !