Global existence for the 2D Kuramoto-Sivashinsky equation

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Outline

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The Kuramoto-Sivashinsky equation

Model for long wavelength instabilities in dissipative systems (e.g., flame front propagation, reaction-diffusion equations).

Study the problem in a 2D periodic box with sides L_1, L_2, identified with a 2D torus $\mathbb{T}^2$.

Integral form: $\phi : \mathbb{T}^2 \times [0, T) \to \mathbb{R}$,

$$\phi_t + \Delta^2 \phi + \Delta \phi + \frac{1}{2} |\nabla \phi|^2 = 0 \qquad \text{(KSE)}$$

Derivative form: $\mathbf{u} : \mathbb{T}^2 \times [0, T) \to \mathbb{R}^2$, $\mathbf{u} = \nabla \phi$,

$$\partial_t \mathbf{u} + \Delta^2 \mathbf{u} + \Delta \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2 = 0, \qquad \text{curl } \mathbf{u} = 0.$$

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The equation cont.

Dissipative system with Burgers (for **u**) or conservation-law (for ϕ) nonlinearity and instability at large scales.

Even in 1D non-trivial, long-time dynamics (chaotic trajectories)

Challenges in analysis in 2D:

1. Unstable modes for linearized operator if any period $L_i > 2\pi$. Unstable (generalized) modes in $\mathbb{R}^2$.
2. No maximum principle (biharmonic op.), no *a priori* L^∞ bounds;
3. No energy identity ($\int \mathbf{u} \cdot \nabla |\mathbf{u}|^2 \, dx \neq 0$) for **u** or ϕ, no *a priori* L^2 bounds.

No known *a priori* norm bound (mean of **u** preserved).

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Previous results

1D KS is well understood, global well-posedness. For **2D KS** many basic open problems.

Known results (stated for ϕ):

1. Local-in-time well-posedness for $\phi \in L^p$ (Biswas-Swanson), estimates on determining modes and size of attractor, *assuming* global H^1 bound (Nikolaenko-Scheuer-Temam);
2. Continuation criteria based on H^1 norm (Bellout-Benachour-Titi);
3. Global-in-time well posedness for thin domains (Sell-Taboada, Molinet, Benachour-Kukavica-Rusin-Ziane, Massatt-Kukavica), small data in H^1 or Wiener algebra $\mathcal{B}^1$, one slightly growing mode in each direction (Ambrose-M.);
4. Analyticity and Gevrey regularity (rough data) for $t > 0$ (Ambrose-M., Biswas-Swanson, Stanislavova-Stefanov) in a strip.

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Our results

Work with the integrated form of KSE.

- No growing modes ($L_1, L_2 < 2\pi$): Global-in-time existence of mild solution for small data in L^2.

  Results can be extended to L^p, $1 < p < \infty$.

- Growing modes (L_1 or $L_2 \geq 2\pi$): Global-in-time existence for large data data in L^2, if linear advection by **mixing** or **shear** flow added:

$$\partial_t \phi + \mathbf{v} \cdot \nabla \phi + \frac{1}{2} |\nabla \phi|^2 = -\Delta^2 \phi - \Delta \phi, \qquad \textbf{(AKSE)}$$

$\mathbf{v}$ a given, possibly time-dependent, divergence-free vector field.

AKSE model passive flame propagation in premixed-combustion.

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Evolution of the mean

Set $\bar{\phi}(t) = \fint_{\mathbb{T}^2} \phi(x,t)\, dx$. Let $\psi := \mathbb{P}(\phi) = \phi - \bar{\phi}$.

$\mathbb{P}$ is an orthogonal projection in L^2, bounded projection in L^p, H^s, $s > 0$, and commutes with all Fourier multipliers.

Denote $\mathring{L}^p(\mathbb{T}^2) = \mathbb{P}(L^p(\mathbb{T}^2))$, $\mathring{H}^s(\mathbb{T}^2) = \mathbb{P}(H^s(\mathbb{T}^2))$, $s > 0$.

Norm in $\mathring{H}^s(\mathbb{T}^2)$ is equivalent to the seminorm in $\dot{H}^s(\mathbb{T}^2)$.

From KSE, AKSE, it follows that:

$$\frac{d}{dt}\bar{\phi} = -\frac{1}{2L_1L_2}\|\nabla\phi\|^2_{L^2} = -\frac{1}{2L_1L_2}\|\nabla\psi\|^2_{L^2}.$$

$\Rightarrow$ have control on $\bar{\phi}$ on $[0,T]$ if $\psi \in L^2(0,T;L^2(\mathbb{T}^2))$.

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Mild formulation

Set $\psi(t)(\mathbf{x}) = \psi(\mathbf{x}, t)$ and $\psi_0 = \psi(0)$.

Say ψ is a *mild solution* if

$$\psi(t) = \mathcal{T}_{\psi_0}(\psi)(t) := e^{-t\mathcal{L}}\psi_0 + B(\psi, \psi)(t) + L(\psi)(t),$$ where

1. Linearized operator: $\mathcal{L} := \Delta^2 + \Delta$, solution operator $e^{-t\mathcal{L}}$, $t > 0$.
2. Bilinear form:
$$B(\psi_1, \psi_2) := -\frac{1}{2}\int_0^t \mathbb{P}e^{-(t-\tau)\mathcal{L}}\nabla\psi_1(\tau)\cdot\nabla\psi_2(\tau)\,d\tau,$$
3. Linear advection:
$$L(\psi) := -\int_0^t e^{-(t-\tau)\mathcal{L}}\mathbb{P}(v(\tau)\cdot\nabla\psi(\tau))\,d\tau.$$

Seek a solution as a fixed point of the map $\mathcal{T}_{\psi_0} = \mathcal{T}$.

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Properties of $e^{t\mathcal{L}}$

$\mathcal{L}$ generates an (unbounded) analytic semigroup $e^{-t\mathcal{L}}$ on L^p, $1 < p < \infty$.

No growing modes - exponential stability:

1. $\forall T_1 > 0$, $\exists\, \gamma_1, \beta > 0$ such that

$$\|e^{-t\mathcal{L}} f\|_{\mathring{L}^2} \leq \gamma_1\, h_1(t)\, \|f\|_{\mathring{L}^1}, \qquad \forall t > 0,$$

where

$$h_1(t) = \begin{cases} t^{-1/4}, & 0 < t \leq T_1, \\ t^{-1/2}\, e^{-\beta t}, & t > T_1. \end{cases}$$

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No advection- no growing modes: small data in L^2

Assume $L_1, L_2 < 2\pi$, $\mathbf{v} = 0$.

Define the adapted space:

$$\mathbf{X}_\infty := \{\mathbf{f} : [\mathbf{0}, \infty) \times \mathbb{T}^\mathbf{2} \mid \sup_{\mathbf{t}>\mathbf{0}} \mathbf{t}^{\mathbf{1/4}} \|\nabla \mathbf{f}\|_{\mathbf{L^2}} < \infty\}.$$

Let $\widetilde{X}_\infty = C([0,\infty); \mathring{L}^2) \cap X_\infty$ with induced norm:

$$\|f\|_{\widetilde{X}_\infty}. := \mathrm{Max}(\sup_{t\geq 0} \|f\|_{L^2}, \sup_{t>0} t^{1/4} \|\nabla f\|_{L^2}).$$

From the semigroup estimates:

$$B : \widetilde{X}_\infty \times \widetilde{X}_\infty \to \widetilde{X}_\infty,$$

and there exists $\eta > 0$ such that:

$$\|B(\psi_1, \psi_2)\|_{\widetilde{X}_\infty} \leq \eta \, \|\psi_1\|_{\widetilde{X}_\infty} \, \|\psi_2\|_{\widetilde{X}_\infty},$$

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No growing mode, no advection: global existence cont.

Theorem

Let $\psi_0 \in \mathring{L}^2(\mathbb{T}^2)$. $\exists\, \delta > 0$ *such that, if* $\|\psi_0\|_{\mathring{L}^2} < \delta$, $\exists$ *a mild solution* ψ *of the projected KSE in* $\widetilde{X}_\infty$ *such that* $\psi(0) = \psi_0$.

1. Proof is by Banach Contraction Theorem in a suitable ball $B(0, M) \subset \widetilde{X}_\infty$.
2. Solution is unique in $\widetilde{X}_\infty$.
3. Smallness of data is used to control linear trend $e^{t\mathcal{L}}\psi_0$.

If one sightly growing mode present in each direction, can essentially separate evolution of growing modes from remainder.

Growing modes controlled via a **Lyapunov function** argument.

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Dissipation time

Consider the hyperdiffusion-advection equation:

$$\partial_t f + \mathbf{v} \cdot \nabla f + \Delta^2 f = 0.$$

Denote the associated *evolution system* by $S_{s,t}$, $0 \leq s \leq t$.

The number $\tau^* \geq 0$, where

$$\tau^* = \inf \left\{ \mathbf{t} \geq \mathbf{0} \;\middle|\; \|\mathcal{S}_{\mathbf{s},\mathbf{s}+\mathbf{t}}\|_{\mathbf{L}^2 \to \mathbf{L}^2} \leq \frac{\mathbf{1}}{\mathbf{2}}, \text{ for all } s \geq 0 \right\},$$

is called the **dissipation time** associated to the system $\mathcal{S}_{s,t}$, $s \leq t$.

One has $0 < \tau^* < \infty$. τ^* depends on $\mathbf{v}$ and $\tau^*(\mathbf{v}) \leq \tau^*(\mathbf{0})$.

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Enhanced dissipation

Call $\tau^*(\mathbf{v})$ the *dissipation time of (flow of)* $\mathbf{v}$.

Assume $\mathbf{v} \in L^\infty([0,\infty); W^{1,\infty}(\mathbb{T}^2))$.

Study whether $\exists$ flows with velocity $\mathbf{v}$ for which $\tau^*(\mathbf{v}) < \tau^*(\mathbf{0})$.
Seek examples where τ^* can be made *arbitrarily* small.

Flow of $A\,\mathbf{v}$, $A > 0$ amplitude, is said to be **relaxation enhancing** if $\tau^*(A\,\mathbf{v}) \to 0$ as $A \to \infty$.

Examples:

1. Spectral characterization for steady flows (Constantin-Kiselev-Ryzhik-Zlatǒs for Δ);
2. Weakly mixing C^2 flows (informally, $f \circ \Phi^{-1} \rightharpoonup 0$, Φ flow of $\mathbf{v}$).

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More on enhanced dissipation

Say that flow of **v** **mixes with rate** h if, for all $f \in \mathring{H}^1(\mathbb{T}^2)$,

$$\|f \circ \Phi^{-1}(\cdot, t)\|_{\dot{H}^{-1}} \leq h(t)\, \|f\|_{\mathring{H}^1}.$$

Mixing enhances dissipation by transfering energy to *small* scales.

Examples:

1. deterministic examples of **exponentially** mixing flows with $W^{1,p}$-regolarity, $1 \leq p \leq \infty$ (Alberti-Crippa-M., Elgindi-Zlatŏs, Yao-Zlatŏs);
2. random generic examples of **exponentially** mixing flows, smooth in space (Bedrossian-Blumenthal-Punshon Smith).

Flows can enhance dissipation without being mixing (for special data):

1. Certain **cellular** flows (Iyer-Xu-Zlatŏs);
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2. random generic examples of **exponentially** mixing flows, smooth in space (Bedrossian-Blumenthal-Punshon Smith).

Flows can enhance dissipation without being mixing (for special data):

1. Certain **cellular** flows (Iyer-Xu-Zlatŏs);
2. Certain **shear** flows, by hypocoercivity (Albritton-Beekie-Novack, Bedrossian-Coti Zelati, Elgindi, Vicol, ... mostly for Δ).

Effects of enhanced dissipation

Enhanced dissipation can also be measured in terms of decay rates in the **diffusion coefficient** ν.

• It may prevent finite-time blow-up due to concentration, e.g. in aggregation-diffusion (Keller-Segel) models (He-Kiselev, Hopf-Rodrigo, Kiselev-Xu).

• It may stabilize the flow, c.f. inviscid damping for Euler (Bedrossian-Masmoudi, Bedrossian-Coti Zelati).

• It may prevent phase separation, e.g. in Cahn-Hilliard (Feng-Iyer).

For AKSE, we show that small diffusion time leads to global existence with growing modes and large data.

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For AKSE, we show that small diffusion time leads to global existence with growing modes and large data.

Gobal existence with mixing

Theorem

Let $\phi(0) = \phi_0 \in L^2(\mathbb{T}^2)$. Then, there exists a mild solution ϕ of AKSE on $[0, \infty)$, which satisfies:

$$\|\phi(t)\|_{L^2} \leq C_1, \qquad t \geq 0,$$

where C_1 depends only on ϕ_0, provided the dissipation time of $\mathbf{v}$ is small enough.

The bound on τ^* can be made explicit in terms of the size of the initial data.

By contrast, the L^2-norm of the solution to the linearized KSE without advection can grow exponentially.

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Strategy of the proof

- Prove short-time existence of mild solutions ψ of the projected AKSE with data in $\mathring{L}^2$ (same as for KSE).
- Establish a continuation principle based on the L^2 norm (same for KSE).
- Prove that ψ is also a *weak* solution in $L^2(0, T; \mathring{H}^2(\mathbb{T}^2))$, $\forall T > 0$, satisfying an energy inequality (same for KSE).
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Key Lemma: exponential decay of $\|\psi(t)\|_{L^2}$

- Let $B := \|\psi_0\|_{L^2}$. Fix $\mu > 0$.

  Define, with C the constant in the energy inequality:

$$T_0(B) = \int_{B^2}^{4B^2} \frac{1}{Cy + Cy^3}\, dy,$$

$$T_1(B) = \frac{1}{4C(2\mu + 4C + 64CB^4)B + 4C(2\mu + 4C + 64CB^4)^{1/2}}.$$

- On $t_0 \leq t \leq t_0 + T_0(B)$, $0 \leq t_0 \leq T$, $\|\psi(t)\|_{L^2}$ can at most double.
- If dissipation alone is large enough for $0 < \tau < T_0(B)$,

$$\frac{1}{\tau}\int_{t_0}^{t_0+\tau} \|\Delta\psi(t)\|_{L^2}^2\, dt \geq 2\mu\|\psi(t_0)\|_{L^2}^2 + 4C\|\psi(t_0)\|_{L^2}^2 + 64C\|\psi(t_0)\|_{L^2}^6,$$

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Key Lemma cont.

- If dissipation alone is not large enough, then τ^* must be small enough:

$$\tau^* \leq \min\left(T_0(B),\ T_1(B),\ \frac{1}{4\mu}\right).$$

$\Rightarrow$ still have $\|\psi(t_0+\tau)\|_{L^2} \leq e^{-\mu\tau}\|\psi(t_0)\|_{L^2}$, $0 < \tau < \tau^*$.

- Conclude by dividing the interval $[0, T]$, $T > 0$, into subintervals of length τ.

If $\mathbf{v}$ is a **shear** flow ($\mathbf{v}(x, y) = (u(y), 0)$), expect global existence if growing modes only along shear ($0 < L_2 < 2\pi$):

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Global existence with advection by a shear flow

Consider 2D KSE with advection by a **shear flow v** $= A\,(u(y), 0)$:

$$\partial_t \phi + u(y)\partial_x \phi + \frac{\nu}{2}|\nabla \phi|^2 + \nu \Delta^2 \phi + \nu \Delta \phi = 0,$$

where $\nu = A^{-1}$, on the torus $\mathbb{T}^2 = [0, L_1] \times [0, L_2]_{per}$, $0 < L_2 < 2\pi$.

Given $g \in L^2(\mathbb{T}^2)$, we denote

$$\langle g \rangle(y) = \frac{1}{L_1} \int_{\mathbb{T}^1} g(t, x, y)\mathrm{d}x, \qquad g_{\neq}(x, y) = g(x, y) - \langle g \rangle(y).$$

$\langle g \rangle$ projection onto the kernel of the advection operator $u(y)\partial_x$,
$g_{\neq}$ projection onto the orthogonal complement in L^2.

Refer to $\langle \phi \rangle$ and $\phi_{\neq}$ as the *kernel* and *projected components*.

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Projected equations

$\langle\phi\rangle$ satisfies

$$\partial_t\langle\phi\rangle + \frac{\nu}{2L_1}\int_{\mathbb{T}^1}|\nabla\phi_{\neq} + \nabla\langle\phi\rangle|^2\,\mathrm{d}x + \nu\partial_y^4\langle\phi\rangle + \nu\partial_y^2\langle\phi\rangle = 0,$$

while $\phi_{\neq}$ satisfies

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Set $\psi = \partial_y\langle\phi\rangle$. Then

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Pseudo-spectral properties and enhanced dissipation

Let $(X, \|\cdot\|)$ be a complex Hilbert space.
Let H be a closed, densely defined operator on X.

If H is an m-accretive operator on X, then decay of the semigroup e^{-tH} depends on (Wei '18):

$$\Psi(H) = \inf\{\|(H - i\lambda)g\| : g \in D(H),\ \lambda \in \mathbb{R},\ \|g\| = 1\}.$$

Set $\quad H_{\nu,k} := \nu\Delta_k^2 + iku(y),\ \Delta_k := -k^2 + \partial_{yy}$. Then:

$$\|\mathrm{e}^{-H_{\nu,k}t}\|_{\mathrm{op}} \leq \mathrm{e}^{-t\Psi(H_{\nu,k})+\pi/2}, \qquad \forall t \geq 0,$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm.

Decay $H_{\nu,k}$ gives decay of $H_\nu := \nu\Delta^2 + u(y)\partial_x$.

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Pseudo-spectral property cont.

Assume the following condition on the shear (after Gallay):

Assumption

There exist $m, N \in \mathbb{N}$, $c_1 > 0$ and $\delta_0 \in (0, L_2)$ with the property that, for any $\lambda \in \mathbb{R}$ and any $\delta \in (0, \delta_0)$, there exist $n \leq N$ and points $y_1, \ldots y_n \in [0, L_2)$ such that

$$|u(y) - \lambda| \geq c_1 \left(\frac{\delta}{L_2}\right)^m, \qquad \forall\, |y - y_j| \geq \delta, \quad \forall j \in \{1, \ldots n\}.$$

Example: $u(y) = \sin(y)^m$.

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Enhanced dissipation

Proposition

Let u satisfy the Assumption. Assume $k \neq 0$ and $\nu|k|^{-1} \leq 1$. There exists $\varepsilon_0' > 0$, independent of ν and k, such that

$$\Psi(H_{\nu,k}) \geq \varepsilon_0' \, \nu^{\frac{m}{m+4}} \, |k|^{\frac{4}{m+4}}.$$

Corollary

Let P_k be L^2 projection onto the k-th horizontal mode. Then

$$\|e^{-H_\nu t} P_k\|_{op} \leq e^{-\varepsilon_0' \, \nu^{\frac{m}{m+4}} \, |k|^{\frac{4}{m+4}} t + \pi/2}, \qquad \forall t \geq 0,$$

$$\Rightarrow \qquad \|e^{-H_\nu t}\|_{op} \leq e^{-\lambda_\nu' t + \pi/2}, \qquad t > 0,$$

where $\lambda_\nu' = \varepsilon_0' \nu^{\frac{m}{m+4}}$.

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Global existence with shear

Main Result

Let $\phi_0 \in L^2(\mathbb{T}^2)$, $0 < L_2 < 2\pi$, and let $u : [0, L_2) \to \mathbb{R}$ satisfy the Assumption.
Then there exists $0 < \nu_0 < 1$ depending on $L_1,\ L_2, u$ and $\|\phi_0\|_{L^2}$ such that for any $0 < \nu < \nu_0$, there is a global weak solution ϕ of AKSE with initial data ϕ_0.

Theorem extends to u with a finite number of critical points of order at most $m \geq 2$, but with a worse decay rate λ_ν for the semigroup.

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Bootstrap

Local existence and energy estimates imply (cf. Bedrossian-He ’17):

Bootstrap assumptions

For small $t > 0$ and $0 \leq s \leq t$,

1. $\|\phi_{\neq}(t)\|_{L^2} \leq 8e^{-\lambda_\nu t/4}\|\phi_{\neq}(s)\|_{L^2}$,
2. $\nu \int_s^t \|\Delta\phi_{\neq}(\tau)\|_{L^2}^2 \, d\tau \leq 4\|\phi_{\neq}(s)\|_{L^2}^2$.

Let $t_0 > 0$ be the *maximal* time such that the estimates hold on $[0, t_0]$.

$\Rightarrow$ $\exists$ ν-independent $C_1 = C_1(\|\phi_{\neq}(0)\|_{L^2}, \|\psi(0)\|_{L_y^2})$ such that on $[0, t_0]$

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Proof of Main Result:

1. By continuation in L^2 and Lemma, $t_0 = \infty \Rightarrow$ $\phi_{\neq} \in L^\infty([0,\infty); L^2(\mathbb{T}^2)) \cap L^2([0,\infty); H^2(\mathbb{T}^2))$.
2. Hence $\psi = \partial_y \langle\phi\rangle \in L^\infty([0,T); L^2(\mathbb{T}^1)) \cap L^2([0,T); H^1(\mathbb{T}^1)) \Rightarrow$ $\bar{\phi} \in L^\infty([0,T)), \forall\, 0 < T < \infty$.
3. By Poincaré + triangle inequality, $\langle\phi\rangle \in L^\infty([0,T); L^2(\mathbb{T}^1))$ and $\phi = \langle\phi\rangle + \phi_{\neq} \in L^\infty([0,T); L^2(\mathbb{T}^2))$.
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2. $\nu \int_s^t \|\Delta\phi_{\neq}(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leq 2\|\phi_{\neq}(s)\|_{L^2}^2$.

Proof of Main Result:

1. By continuation in L^2 and Lemma, $t_0 = \infty \;\Rightarrow$ $\phi_{\neq} \in L^\infty([0,\infty); L^2(\mathbb{T}^2)) \cap L^2([0,\infty); H^2(\mathbb{T}^2))$.
2. Hence $\psi = \partial_y \langle\phi\rangle \in L^\infty([0,T); L^2(\mathbb{T}^1)) \cap L^2([0,T); H^1(\mathbb{T}^1)) \;\Rightarrow$ $\bar{\phi} \in L^\infty([0,T))$, $\forall\, 0 < T < \infty$.
3. By Poincaré + triangle inequality, $\langle\phi\rangle \in L^\infty([0,T); L^2(\mathbb{T}^1))$ and $\phi = \langle\phi\rangle + \phi_{\neq} \in L^\infty([0,T); L^2(\mathbb{T}^2))$.
4. Finally $\nabla^2\phi = \nabla^2\phi_{\neq} + \nabla\psi \in L^2([0,T); L^2(\mathbb{T}^2))$.

Proof of main result

Lemma-Bootstrap estimates

If $\nu_0 > 0$ small enough and $\nu < \nu_0$, then for all $0 \leq s \leq t \leq t_0$,

1. $\|\phi_{\neq}(t)\|_{L^2} \leq 4\mathrm{e}^{-\lambda_\nu t/4}\|\phi_{\neq}(s)\|_{L^2}$,
2. $\nu \int_s^t \|\Delta\phi_{\neq}(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq 2\|\phi_{\neq}(s)\|_{L^2}^2$.

Proof of Main Result:

1. By continuation in L^2 and Lemma, $t_0 = \infty \;\Rightarrow$ $\phi_{\neq} \in L^\infty([0,\infty); L^2(\mathbb{T}^2)) \cap L^2([0,\infty); H^2(\mathbb{T}^2))$.
2. Hence $\psi = \partial_y\langle\phi\rangle \in L^\infty([0,T); L^2(\mathbb{T}^1)) \cap L^2([0,T); H^1(\mathbb{T}^1)) \;\Rightarrow$ $\bar{\phi} \in L^\infty([0,T))$, $\forall\, 0 < T < \infty$.
3. By Poincaré + triangle inequality, $\langle\phi\rangle \in L^\infty([0,T); L^2(\mathbb{T}^1))$ and $\phi = \langle\phi\rangle + \phi_{\neq} \in L^\infty([0,T); L^2(\mathbb{T}^2))$.
4. Finally $\nabla^2\phi = \nabla^2\phi_{\neq} + \nabla\psi \in L^2([0,T); L^2(\mathbb{T}^2))$.

THANK YOU!