Foam evaluation, link homology ② and
① Soergel bimodules ③

jt with E. Wagner and M. Khovanov.

1702.04140

1801.02244

+ on-going work.

① Aim a) define a category

b) give and explain a formula

Cat FOAM.

ob: MOY

mor: foa

Def: A M

- trivalent
- oriented
- labeled.
- plane

graph.

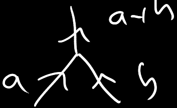
graph.

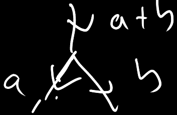
ny. co}.

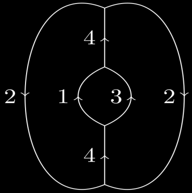
OY graph Γ is

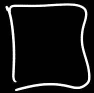
allowed ↪ $\mathbb{Q}i$

$l : E(\Gamma) \to \mathbb{Z}_{>0}$] flow condit

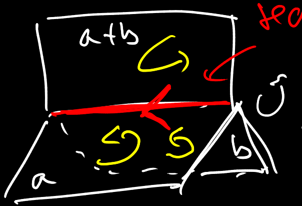








disk



Def: A (cl

is a finite

with 3 loc

$s(F) = \{\text{seams}\} \cup \{\text{sing. Vx}\}$.

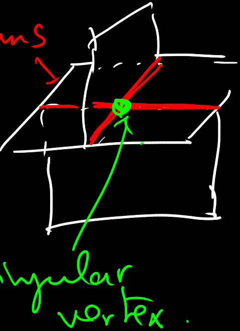
facets

= Conn. comp of F \

osed) foam

CW complex.

cal models


ms
ingular vertex.

S(F).

- labelling of facets:

$l : \{\text{facets}\} \to \mathbb{Z}_{>0}.$ } flow cond

orientations of facets and seams

- cyclic ordering of facets around seams

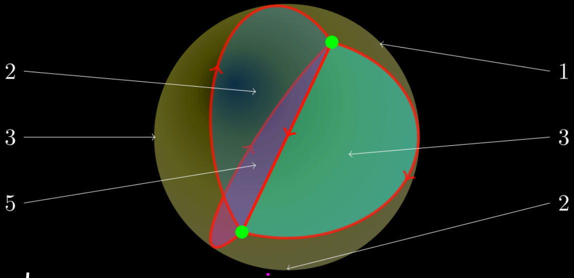
- singular vertices are emb. i

$\mathbb{R}^3$ with right rule

-tions

Each facet carries a sym pol. in $\ell(f)$ variables

$$P_f \in \mathbb{Z}[t_1, \dots t_{\ell(f)}]^{S_{\ell(f)}}.$$



n

Define foam with boundary.

and cob ... ⇝ FOAM.

b) fix N

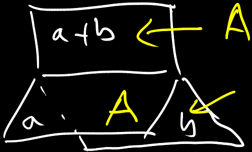
F foam

Def: an

is a mc

$P(\{X_1, \dots$

$\# c(f) =$



$) \in \mathbb{Z}_{\geq 0}.$

N-coloring of F

p $c : \{ \text{facets} \} \longrightarrow$

$\ldots, X_N \}) $ s.t.

$c(f)$ + flow condition

$\cup B$

B

Foam evaluation formula

fix F foam c coloring

$$\tau_N(F,c) = \frac{(-1)^{\sum_{i=1}^{N} i\chi(F_i(c))/2 + \sum_{1\leq i<j\leq N} \theta_{ij}^{+}(F(c))} \prod_f P_f}{\prod_{1\leq i<j\leq N} (X_i - X_j)^{\frac{\chi(F_{ij}(c))}{2}}}$$

χ is Euler characteristic.

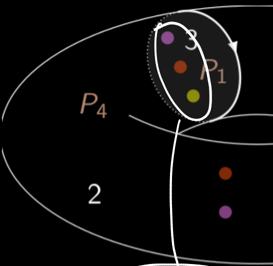
$$\left.\begin{aligned} F_i(c) &= \bigcup_{f \text{ s.t. } X_i \in c(f).} f \\ F_{ij}(c) &= \overline{F_i \,\Delta\, F_j} \end{aligned}\right]$$

both are orientable

$(c(f))$

$\theta^+_{i,j}$: using cyclic orientation assign signs to these circles

$\theta^+_{i,j}$ = # positive circles.



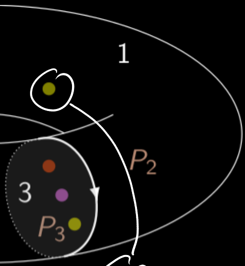
$P_1 = t_1 + t_2 + t_3$

$P_3 = 1$

$$\frac{(-1)^{1+2+4}(X_1 + X_2 +}{\underbrace{(X_1 - X_3)}(X_2}$$

	X_1X_2	X_1X_3
$\chi(F_{\bullet\bullet})$	0	2
$\theta^+_{\bullet\bullet}(F)$	2	0

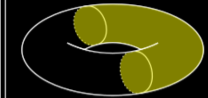
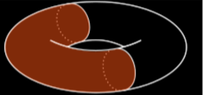
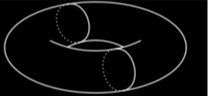
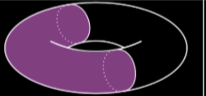
$\tau_N(F) =$



$P_2 = t_1^2$

$P_4 = t_1 t_2 (t_1 + t_2)$

$$\frac{X_4)X_1^2(X_2X_4(X_2+X_4))}{-X_3)(X_3-X_4)}$$

ℙ	Monochrome	$\chi(F_\bullet)$
X_1		2
X_2		2
X_3		0
X_4		2

	X_1X_4	X_2X_3	X_2X_4	X_3X_4
	0	2	0	2
	2	0	0	0

$N = 4$

$\mathbb{P} = \{X_1, X_2, X_3, X_4\}.$

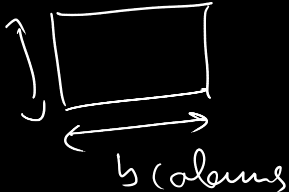
$$\sum_c \tau_N(F,c)$$

Prop: $\tau(F) \in \mathbb{Z}[X_1, \dots X_N]^{S_N}.$

Prop: α, β, λ are You

diagrams fittin

in



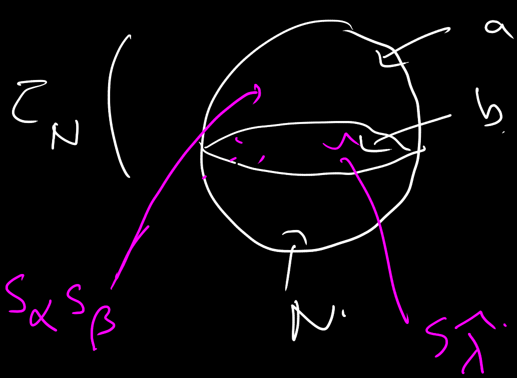
Then:

with $N = a + b.$

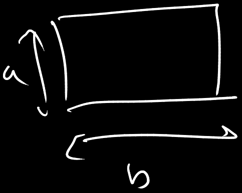
ng

g

$$c^{\lambda}_{\alpha\beta} = (-1)^{?}\, Z_N\Big($$


a
b
$S_\alpha S_\beta$
N.
$S_{\hat\lambda}$.

$\hat\lambda$ si transpose of compl. of λ in



II - Link homology :

Γ MOY graph.

$$\begin{tikzcd} V_N(\Gamma) = \bigoplus_{F:\emptyset \to \Gamma} R_N & \mathbb{Z}[X_1 \dots X_N]^{S_N} \arrow[l] \end{tikzcd}$$

$$\langle , \rangle_N : V_N(\Gamma)^{\otimes 2} \longrightarrow R_N.$$

$$F \otimes G \longmapsto \tau_N(F \cup_\Gamma \bar{G}) \in R_N.$$

$$\mathcal{F}_N(\Gamma) = V_N(\Gamma) / \mathrm{Ker} \langle , \rangle_N$$

FOAM

Prop: $\mathcal{F}_N$

$\downarrow$

free

$$\begin{matrix}\boxed{P} \\ \circ \\ \boxed{Q}\end{matrix} \longrightarrow$$

Lemma: $\mathcal{F}_N$ induces a functor:

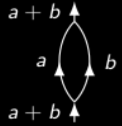
$\longrightarrow R_N\text{-mod}.$

$$\underbrace{\left(\bigcirc_k\right)}\simeq H^*_T(Gr_{\mathbb{C}}(k,N))$$

R_N-module of rank $\begin{bmatrix} N \\ k \end{bmatrix}$

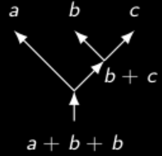
$\boxed{PQ}$

$\mathcal{F}_N\Bigg($



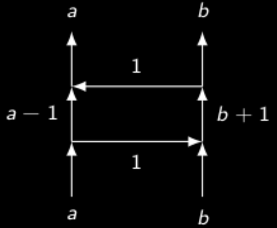
$\Bigg)\simeq$

$\mathcal{F}_N\Bigg($



$\Bigg)\simeq$

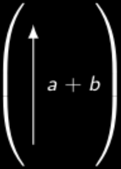
$\mathcal{F}_N\Bigg($


a
b
1
a − 1
b + 1
1
a
b

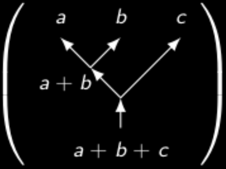
$\Bigg) =$

$\oplus$

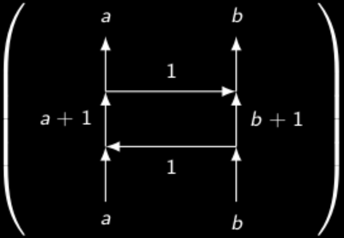
$\begin{bmatrix}a+b\\a\end{bmatrix}\mathcal{F}_N$



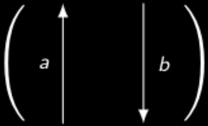
$\mathcal{F}_N$



$\mathcal{F}_N$


a
b
1
a + 1
b + 1
1
a
b

$[b-a]\mathcal{F}_N$



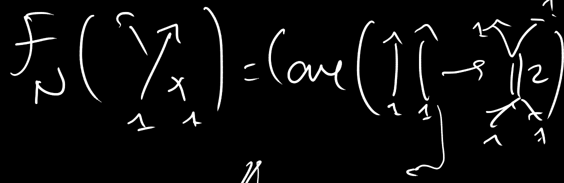
Using these iso we understand the value of $\mathcal{F}_N$ on all MOY graphs.

"Cor" (Khovanov-Rozansky, Lauda-Queffelec-Rose, Mackaay-Stošic-Vaz

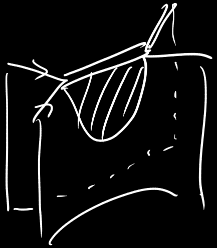
$\mathcal{F}_N$ can be extended to a link homology theory by

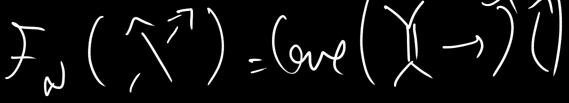
-)

defining:



F_N
x
1
1
= Cone
1
1
1
2
1
1





"Cor" (Ehrig - Tubbenhauer - Wedrich)

These homologies are functorial

III

Ex

C

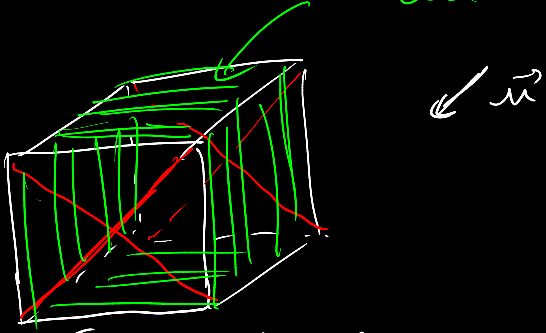
a

ab

n

:- Soergel bimodules

x a cube ∂ allowed.



=

face F is disk like if

⊆ C

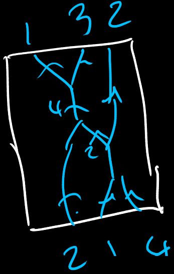
at point $x \in F$, $(\vec{n}_x, \vec{u}) > 0$

illebaille

bicat: ob: finite s

k

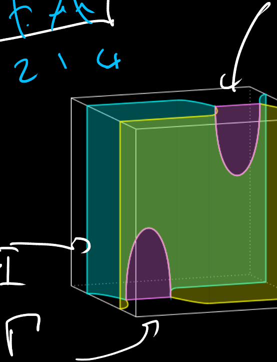
1-mor: direct MOY



2-mor:

$\Gamma \to \Gamma'$

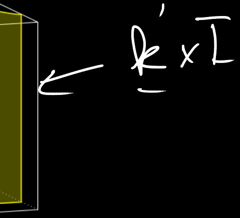
$k \to k'$, $k \times I$



ea of integers > 0.

graph.

Γ'


$\underline{k}' \times I$

Def: Two formal linear sums of disk like foam with same bdy

$$\sum a_i F_i \ , \ \sum b_j G_j$$

are ∞-equivalent if:

$\forall N$ and all foam H with same bdy.

$$\sum a_i \tau_N(F_i \cup \bar{H}) = \sum b_j \tau_N(G_j \cup \bar{H}).$$

Γ

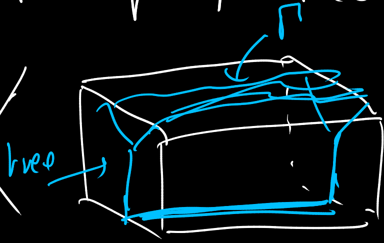
Thm (Khovanov-R.-Wagner):

$R : 2\mathrm{FoAM} \longrightarrow \mathrm{Soergel}.$

$\underline{k} \longmapsto A_{\underline{k}} = \mathbb{Q}[x_1, \dots x_k]^{S_{\underline{k}}}$

$\Gamma \longmapsto \mathrm{Soergel}(\Gamma)$

$F \longmapsto$ Morphism

$\longmapsto \Big\langle$  $\Big\rangle_{\mathbb{Q}} \Big/ \infty.$

Prop

Q

and

of

: R factorizes by

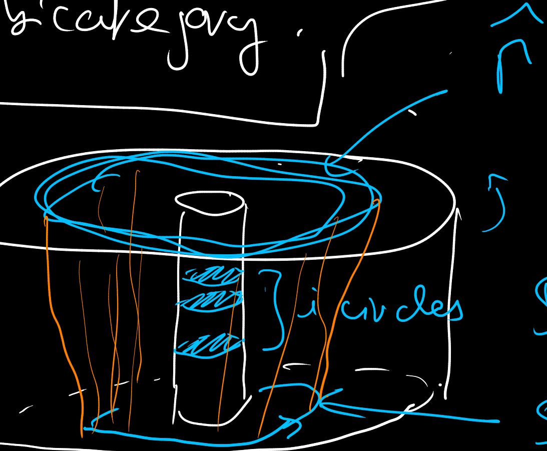
FoAn / ∞-eq

induces equivalence

bicategory.

$HH_i(\mathrm{Serpe}(\Gamma))$

$=$


$\hat{r}$
} i circles

$\rangle \;/_{\infty\text{-eq}}\; Q$

$S_1 \circlearrowleft G_1$

$S_{k-i} \circlearrowleft G_{k-i}$