BANFF (Visio)
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Distribution of rational curves

As was explained in previous talks the analogies between rational curves and rational points is a recurrent inspiration to go back and forth from arithmetic geometry to algebraic geometry. So today I would like to use this

dictionnary

rational points	rational curves
$V(\mathbb{Q})$	$\mathrm{Map}(\mathbb{P}^1, V)$

to translate a talk I gave at the Journees arithmétiques in 2001. I should stress that this is a

Work in progress

although it has been so for quite while since the first talk I gave about that was in Hyderabad in 2003. I should also mention that is in part inspired by

Talks of V.V. BATYREV, J. ELLENBERG.

I Examples Let me start with simple examples

k field $\mathcal{M}_k$: Grothendieck ring of varieties $/k$

generators : $[X]$ isomorphism class of variety $/k$

relations $[X] = [U] + [F]$ if $F \underset{\text{closed}}{\subset} V$, $U = V - F$

product $[X][Y] = [X \times Y]$

Tate motive $\mathbb{L} = [\mathbb{A}^1_k]$

1) Projective space

Let us consider the class $[\mathrm{Mor}^d(\mathbb{P}^1, \mathbb{P}^n)] \in \mathcal{M}_k$

and let $W_d = \left\{ (P_0, \dots, P_n) \in k[T]^{n+1} \;\middle|\; \begin{cases} \gcd(P_0, \dots, P_n) = 1 \\ \max(\deg(P_i)) = d \end{cases} \right\}$

$W_d \longrightarrow \mathrm{Mor}^d(\mathbb{P}^1, \mathbb{P}^n)$ is a $\mathbb{G}_m$ torsor

$[W_d] = (\mathbb{L} - 1)\,[\mathrm{Mor}^d(\mathbb{P}^1, \mathbb{P}^n)]$

If $(P_0, \dots, P_n) \in k[T]^{n+1}$, $\max(\deg(P_i)) = d$, $D = \gcd(P_0, \dots, P_n)$

if $d' = \deg(D)$, $\left(\frac{P_0}{D}, \dots, \frac{P_n}{D}\right) \in W_{d-d'}$

We get
$$\mathbb{L}^{(n+1)(d+1)} - \mathbb{L}^{(n+1)d} = \sum_{d'+d''=d} \mathbb{L}^{d'} [W_{d''}]$$

Taking formal series in $\mathcal{M}_k[[T]]$ we get
$$(\mathbb{L}^{n+1}-1) \sum_d \mathbb{L}^{(n+1)d} T^d = \left(\sum_d \mathbb{L}^d T^d\right)\left(\sum_d [W_d] T^d\right)$$
and thus in $\mathcal{M}_k[\mathbb{L}^{-1}]$

$$[\mathrm{Mor}^d(\mathbb{P}^1, \mathbb{P}^n)]\, \mathbb{L}^{-(n+1)d} = \frac{\mathbb{L}^{n+1}-1}{\mathbb{L}-1}(1-\mathbb{L}^{-n}) \text{ if } d \geq 1$$

2) Products of projective space $V = \prod_{i=1}^m \mathbb{P}^{n_i}$ is a trivial consequence, if we take into account the various degrees or the class

$$\deg : \mathrm{Mor}(\mathbb{P}^1, V) \longrightarrow \mathbb{Z}^m = \mathrm{Hom}(\mathrm{Pic}(V), \mathbb{Z})$$
$$[\mathrm{Mor}^{\underline{d}}(\mathbb{P}^1, V)] = \prod_{i=1}^m [\mathrm{Mor}^{d_i}(\mathbb{P}^1, \mathbb{P}^{n_i})]$$

So

$$[\mathrm{Mor}^{\underline{d}}(\mathbb{P}^1, V)]\, \mathbb{L}^{-\sum_{i=1}^m (n_i+1)d_i} = \prod_{i=1}^m \frac{\mathbb{L}^{n_i+1}-1}{\mathbb{L}-1}(1-\mathbb{L}^{-n_i}) \text{ if all } d_i \geq 1$$

N.B. $\omega_V^{-1} = \boxtimes_{i=1}^m \mathcal{O}_{\mathbb{P}^{n_i}}(n_i+1)$

3) $\mathbb{P}^2$ blown up in a point

$$\begin{array}{ccc} V & \hookrightarrow & \mathbb{P}^2 \times \mathbb{P}^1 \\ & & [x:y:t]\ [u:v] \\ \big\downarrow \pi = p_1 & & \\ \mathbb{P}^2 & & \end{array}$$

$yu = xv$

$\mathrm{Pic}(V) \simeq \mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^1) = \mathbb{Z}^2$

The computation can again be done in an elementary way:

$$[\mathrm{Mor}^{\underline{d}}(\mathbb{P}_1, V)]\, \mathbb{L}^{-3d_1-2d_2} = \frac{\mathbb{L}^3-1}{\mathbb{L}-1}(1-\mathbb{L}^{-2}) \times \frac{\mathbb{L}^2-1}{\mathbb{L}-1}(1-\mathbb{L}^{-1}) + E_{d_1,d_2}$$

$\dim(E_{d_1,d_2}) \longrightarrow -\infty$ as $d_1+d_2 \to +\infty$ and $d_2 \to +\infty$

$(\dim([V]\,\mathbb{L}^k) = \dim(V) + k$, for any $k \in \mathbb{Z})$

N.B. $\langle \omega_V^{-1}, \underline{d} \rangle = 3d_1 + 2d_2$

II General setting

1) Aim

- $\mathcal{C}$ smooth proj. geom. integral curve $/k$, $K = k(\mathcal{C})$
  V/K sm, proj, g.i. dim n, $\mathcal{V}$ projective model of V over $\mathcal{C}$.
  Assume V is « almost » Fano
  ω_V^{-1} big, V is rationally connected, ...
  $\mathrm{Pic}(\bar{V})$ free $\mathbb{Z}$-module of finite rank r
  The cone of effective divisors $C_{eff}(\bar{V})$
  is finitely generated.

  Using models of a basis of $\mathrm{Pic}(V)$, get

$$\deg : \underbrace{\mathrm{Mor}_{\mathcal{C}}(\mathcal{C}, \mathcal{V})}_{\text{sections}} \longrightarrow \mathrm{Pic}(V)^\vee = \mathrm{Hom}(\mathrm{Pic}(V), \mathbb{Z})$$

$$\text{movable curves} \longmapsto C_{eff}(V)^\vee = \{ y \in \mathrm{Pic}(V)^\vee_{\mathbb{R}} \mid \forall D \text{ eff}, \langle D, y\rangle \geq 0 \}$$

Get

$$[\mathrm{Mor}^{d}_{\mathcal{C}}(\mathcal{C}, \mathcal{V})] \, \mathbb{L}^{-\langle \omega_V^{-1}, \overset{\text{moveable}}{d} \rangle} \in \mathcal{M}_k[\mathbb{L}^{-1}]$$

We want study the limit of these symbols; for that you need a topological ring. In the examples I mentioned the dimension going to $-\infty$

2) Filtrations on $\mathcal{M}_k[\mathbb{L}^{-1}]$

- dimension

$$F^r_{dim} \mathcal{M}_k[\mathbb{L}^{-1}] = \langle [X]\mathbb{L}^{i}, \dim(X) + i \leq -r \rangle$$

- weight [M. Bilu] (k char. 0)

$$F^r_w \mathcal{M}_k[\mathbb{L}^{-1}] = \mathrm{Hdg}^{-1}(W^r K_0(\mathrm{MHM}))$$

$$\hat{\mathcal{M}}_k = \varprojlim_r \mathcal{M}_k[\mathbb{L}^{-1}] / F^r \mathcal{M}_k[\mathbb{L}^{-1}]$$

3) Motivic Tamagawa volume

M. Bilu defined Euler motivic products in $\hat{\mathcal{M}}_k$.

$\mathcal{X}$ $\quad [\mathcal{X}_p]\mathbb{L}^{-n} = 1 + R_p \quad w(R_p) \leq -3$

$\downarrow \dim_{\mathcal{C}} \mathcal{X} = n$ get $\prod_{p \in \mathcal{C}} [\mathcal{X}_p]\mathbb{L}^{-n} \in \hat{\mathcal{M}}_k$ (leaving some

$\mathcal{C}$ technical points under the rug)

$S \overset{\text{finite}}{\subset} \mathcal{C}$ set of bad points

$\mathcal{C}^\circ = \mathcal{C} - S$

$$\begin{array}{c} \mathcal{T} \\ \downarrow \\ V \end{array} \quad \text{universal torsor : } T_{NS} \text{ torsor } (T_{NS} \text{ alg group})$$

$\overline{T}_{NS} \underset{\text{over } \overline{k}}{\xrightarrow{\sim}} \mathbb{G}_m^r$, $\quad \mathrm{Hom}(\overline{T}_{NS}, \overline{\mathbb{G}}_m) \xrightarrow{\sim} \mathrm{Pic}(\overline{V})$ as galois module

$$\begin{array}{c} \mathfrak{T} \\ \downarrow \\ \mathcal{V}_{\mathcal{C}^\circ} \end{array} \quad \text{model of} \quad \begin{array}{c} \mathcal{T} \\ \downarrow \\ V \end{array}$$

then $w([\mathfrak{T}_p]\mathbb{L}^{-(n+r)} - 1) \leq -3$.

(have to impose extra conditions)

Definition

$$\tau(V) = \sum_{[\mathfrak{T}],\, \mathfrak{T}(K) \neq \emptyset} \mathbb{L}^{n(1-g)} L_S^*(\mathrm{Pic}(\overline{V}), 1) \prod_{p \notin S} ([\mathfrak{T}_p]\mathbb{L}^{-(n+r)})$$

$g = $ genus$(\mathcal{C})$

$$\times \prod_{p \in S} \lim_i [\mathcal{V}(\mathcal{O}_p/\mathfrak{m}_p^i)]\mathbb{L}^{-ni}$$

Remark

Take $\mathcal{V} = V \times \mathcal{C}$

get $\tau(V) = [\mathrm{Jac}(\mathcal{C})]^r \dfrac{\mathbb{L}^{n(1-g)}}{(1-\mathbb{L}^{-1})^r} \displaystyle\prod_{p \in \mathcal{C}} (1-\mathbb{L}^{-1})^r \frac{[V(k(p))]}{\mathbb{L}^n}$

III Guesses

Empirical formula

$$[\mathrm{Mor}^d_{\mathcal{C}}(\mathcal{C}, \mathcal{V})']\mathbb{L}^{-\langle d, \omega_V^{-1}\rangle} \xrightarrow[\substack{d \in C_{\mathrm{eff}}(V)^\vee \cap \mathrm{Pic}(V)^\vee \\ \mathrm{dist}(d, \partial C_{\mathrm{eff}}(V^\vee)) \to \infty}]{} \tau(V)$$

$'$ = moveable

Spirit of the formula

Contravariant functor

$$\{S \text{ dim-0 scheme in } \mathcal{C}\} \longrightarrow k \text{ variety}$$

$$S \longmapsto \mathrm{Hom}_S(S, \mathcal{V}_{|S})$$

$$\mathrm{Mor}_{\mathcal{C}}(\mathcal{C}, \mathcal{V}) \longrightarrow \mathrm{Mor}_S(S, \mathcal{V}_S)$$

$$[\mathrm{Mor}^d_e(\mathcal{C}, \mathcal{V})]\, \mathbb{L}^{-\langle d, \omega_V^{-1}\rangle} \xrightarrow[d\to\infty]{} \text{"}\underline{\lim}\ \mathrm{Hom}_s(\mathcal{S}, \mathcal{V}_s)\text{"}$$

This is related to equidistribution

<u>Equidistribution</u>

$\mathcal{S} \subset \mathcal{C}$, $Y \subset \mathrm{Mor}_{\mathcal{S}}(\mathcal{S}, \mathcal{V})$

$$\text{"}\frac{[\mathrm{Mor}^d_e(\mathcal{C}, \mathcal{V})_Y]}{[\mathrm{Mor}^d_e(\mathcal{C}, \mathcal{V})]} \xrightarrow[d\to\infty]{} \frac{[Y]}{[\mathrm{Mor}_{\mathcal{S}}(\mathcal{S}, \mathcal{V})]}\text{"}$$

restriction to $\mathcal{S}$ belongs to Y

III <u>Results</u> (without equidistribution).

- D. Bourqui & M. Bilu: split toric varieties (equivariant compactification of algebraic tori) (implies all easy examples from the beginning)
- M. Bilu, L. Faisant: equivariant compactification of affine space (using motivic Poisson formula)
- L. Faisant: toric fibrations, which are locally trivial for Zariski topology.

IV <u>Accumulating phenomena</u>

$V \subset \mathbb{P}^4_{\mathbb{C}}$ cubic volume $\quad \mathrm{Pic}(V) = \mathrm{Pic}(\mathbb{P}^4_{\mathbb{C}})$

$S = \{\text{proj. line} \subset V\}$ Fano surface (of general type)

$P \xrightarrow{\ \pi\ } V$ generically degree 6

$$P \xrightarrow{\text{tautological } \mathbb{P}^1 \text{ bundle}} S$$

Among the morphisms from $\mathbb{P}^1$ to V you have

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] \arrow[dr] & V \\ & P \arrow[u, "\pi"'] \end{tikzcd}$$

$\mathrm{Mor}^d(\mathbb{P}^1, P) \subset \mathrm{Mor}^d(\mathbb{P}^1, V)$ not negligible!

(difference of dimensions is, I think, constant)