Toric Kato manifolds

Nicolina Istrati
(joint with A. Otiman, M. Pontecorvo and M. Ruggiero)

University of Marburg

Locally Conformal Symplectic Manifolds: Interactions and Applications
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Plan of talk

1. Motivations

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2. Kato manifolds

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2. Kato manifolds
3. Toric constructions

LCK structures

Definition

- A **locally conformally Kähler (LCK) metric** on a complex manifold X is a Hermitian metric Ω so that

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- LCK structure $\equiv$ global Kähler metric ω on the u.c. $\tilde{X}$ on which $\pi_1(X)$ acts by homotheties

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Q: Is this true for all LCK's? $\rightsquigarrow$ need more higher-dimensional examples

Global spherical shells

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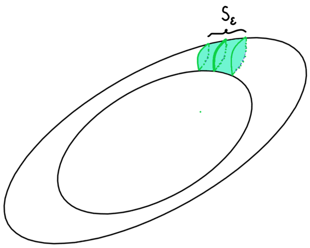

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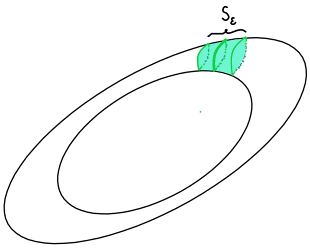
A compact complex manifold with $n \geqslant 2$ containing a GSS is called a **Kato manifold**.



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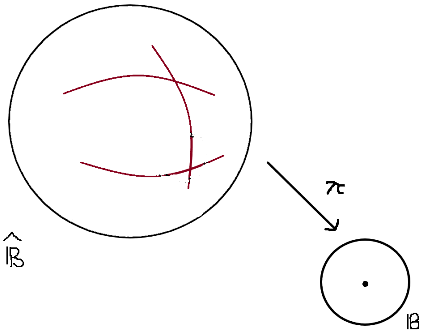

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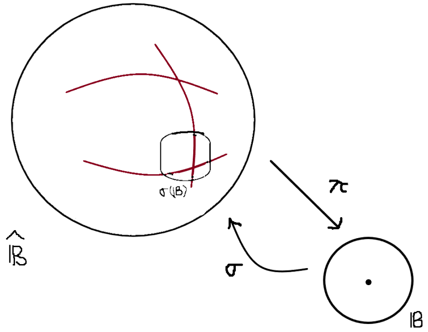

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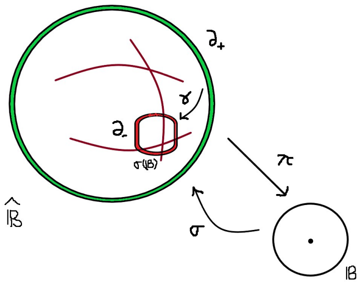

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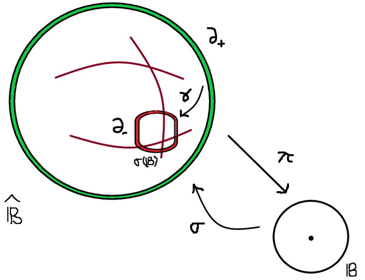

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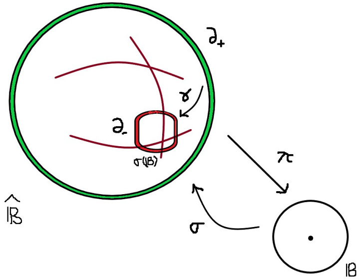

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(π, σ)=**Kato data**

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- $\pi_1(X(\pi, \sigma)) = \mathbb{Z}$

Topological properties

$$\widetilde{X(\pi,\sigma)} = \bigsqcup_{m\in\mathbb{Z}} W_m/_{\partial_- W_m \sim^{\gamma} \partial_+ W_{m+1}}$$

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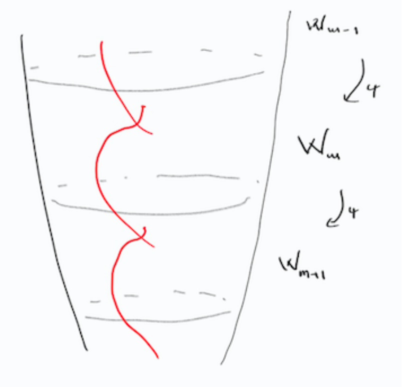

LCK metrics on Kato manifolds

Theorem (Brunella '11; -, Otiman, Pontecorvo, Ruggiero)

A Kato manifold $X = X(\pi, \sigma)$ admits an LCK metric if and only if $\pi : \hat{\mathbb{B}} \to \mathbb{B}$ is Kähler, if and only if $\tilde{X}$ is Kähler.

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- if $\pi \neq \text{id}$, then $X(\pi, \sigma)$ admits no LCK with potential/Vaisman metric

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- Replace initial fundamental domain by $\hat{\mathbb{B}}_r - \sigma(\mathbb{B}_r)$
- Since $\gamma^*(\omega_2|_{\partial_-}) = \frac{c}{C} \cdot \omega_2|_{\partial_+}$ $\rightsquigarrow$ LCK structure

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• Some special cases of toric Kato manifolds were constructed by Tsuchihashi '87

Toric Kato manifolds

Theorem (-, Otiman, Pontecorvo, Ruggiero)

Let X be a toric Kato manifold. Then there exists a uniquely associated toric variety (of non finite type) $X(\tilde{\Sigma})$ and a group $\mathbb{Z} \cong \Gamma \subset \mathrm{Aut}(X(\tilde{\Sigma}))$ so that the universal cover $\tilde{X}$ is a Γ-invariant open subset of $X(\tilde{\Sigma})$ and $X = \tilde{X}/\Gamma$.

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- Take $U := \{z \in (\mathbb{C}^*)^n \mid \exists \lim_{m \to \infty} ||\gamma^m(z)|| = 0\}$

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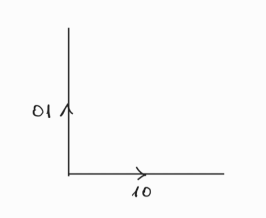
- $A \in \mathrm{Aut}(\tilde{\Sigma}) \;\rightsquigarrow\; F_A \in \mathrm{Aut}(X(\tilde{\Sigma}))$
- $\gamma := \underline{\lambda} \circ F_A \;\rightsquigarrow \Gamma := \langle \gamma \rangle \subset \mathrm{Aut}(X(\tilde{\Sigma}))$
- Take $U := \{z \in (\mathbb{C}^*)^n \mid \exists \lim_{m \to \infty} ||\gamma^m(z)|| = 0\}$
- Then $\tilde{X} = \mathrm{Int}(\overline{U}^{X(\tilde{\Sigma})})$ and $X = \tilde{X}/\Gamma$.

An example - a parabolic Inoue surface

- $\pi : \hat{\mathbb{C}}^2 \to \mathbb{C}^2$ blow-up at 0



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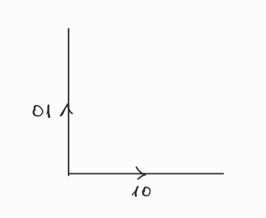

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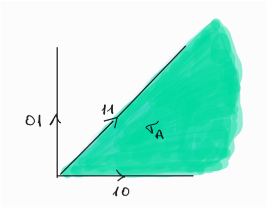

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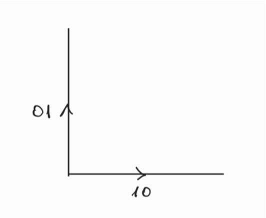

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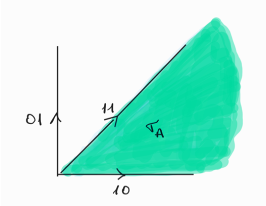

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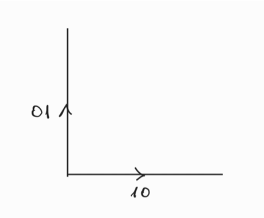

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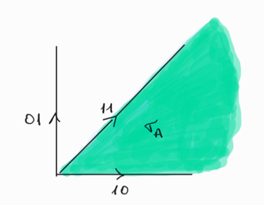

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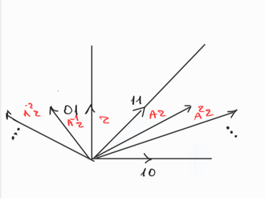

$\tilde{X} = X(\tilde{\Sigma})$

Invariant curves

Let X^n be a toric Kato manifold with matrix A.
Let $P(A) \in \mathrm{GL}(k, \mathbb{Z})$ be the maximal permutation submatrix of A.
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- X is a Hopf manifold iff any T-invariant curve is elliptic;
- X is of hyperbolic type iff any T-invariant curve is rational;
- X is of parabolic type iff X contains a unique T-invariant elliptic curve E and at least one T-invariant rational curve $\rightsquigarrow$ all other T-invariant curves are rational.

Analytic invariants

Theorem (-, Otiman, Pontecorvo, Ruggiero)

We have the following invariants:

$$\mathrm{kod}(X) = -\infty$$

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If moreover X is of hyperbolic type, or of parabolic type with $|\underline{\lambda}|$ is small enough, then:

$$h^{0,0}(X) = h^{0,1}(X) = 1,\ h^{0,p}(X) = 0,\ p \geqslant 2$$
$$h^{1,p}(X) = 0,\ p \neq 1$$
$$h^{1,1}(X) = b_2 = \sharp\{\text{irreducible components of } D\} > 0.$$

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- Degenerate X to a singular, but "easier" space $+$ semi-continuity

Degenerations

Theorem (Nakamura; Tsuchihashi; -, Otiman, Pontecorvo, Ruggiero)

Let X be a toric Kato manifold. Then there exists a flat holomorphic proper family $p : \mathcal{X} \to \Delta$, where $1 \in \Delta \subset \mathbb{C}$ is an open disk, s.t.

- $\mathcal{X}_1 \cong X$
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If moreover X is of hyperbolic type then $\forall t \in \Delta^*, \mathcal{X}_t \cong X$.

Betti numbers

Let X be a toric Kato manifold obtained from $\pi : \hat{\mathbb{C}}^n \to \mathbb{C}^n$. Denote by a_j the number of j-dim cones of the fan of $\hat{\mathbb{C}}^n$.

Theorem (-, Otiman, Pontecorvo, Ruggiero)

X has the following Betti numbers:

$$b_0(X) = b_1(X) = b_{2n-1}(X) = b_{2n}(X) = 1$$

$$b_{2j+1}(X) = 0, \quad 1 \leqslant j \leqslant n-2$$

$$b_{2j}(X) = -1 + \sum_{s=j}^{n} (-1)^{s-j} \binom{s}{j} \left(a_{n-s} + \binom{n}{s+1} \right), \quad 1 \leqslant j \leqslant n-1.$$

In particular, we have

$$b_2(X) = \sharp D, \quad \chi(X) = a_n - 1.$$

Thank you for your attention.