Cohomology of Congruence Subgroups, Steinberg Modules, and Real Quadratic Fields

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Overview

Given a real quadratic field, there is a naturally defined Hecke-stable subspace of the cohomology of a congruence subgroup of $\mathrm{SL}_3(\mathbb{Z})$. We investigate this subspace and make conjectures about its dependence on the real quadratic field and the relationship to boundary cohomology. This is joint work with Avner Ash.

Avner Ash (Boston College)

Tits building

Fix a field K and a positive integer $n \geq 2$.

The *Tits building* $T(K^n)$:

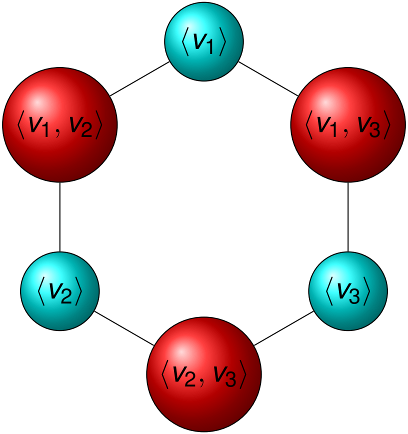
- $(n-2)$-dimensional simplicial complex with one vertex for each subspace $0 \neq V \neq K^n$;
- the vertices $V_1, V_2, \ldots, V_k$ span a simplex if and only if they can be arranged in a flag

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k \subsetneq K^n.$$

$T(K^n)$ is homotopy equivalent to a wedge of $(n-2)$-spheres [Solomon-Tits].

$n = 3$

$v_1, v_2, v_3 \in K^3$, linearly independent:



Steinberg module

The *Steinberg module* $\mathrm{St}(K^n)$ is the reduced homology of the Tits building:

$$\mathrm{St}(K^n) = \widetilde{H}_{n-2}(T(K^n), \mathbb{Q}).$$

- Since $\mathrm{GL}_n(K)$ acts on the Tits building, the Steinberg module is a left module for the group ring $\mathbb{Q}\,\mathrm{GL}_n(K)$.
- The definition can be extended by replacing $\mathbb{Q}$ with any ring R.

Modular Symbols

Theorem (Ash-Rudolph, 1979)

1. *As abelian group,* $\mathrm{St}(K^n;\mathbb{Z})$ *is generated by* $[v_1, v_2, \ldots, v_n]$ *as* $v_1, v_2, \ldots, v_n$ *range over all elements of* K^n.
2. *The following relations hold:*
    1. $[v_1, v_2, \ldots, v_n] = 0$ *if* $v_1, v_2, \ldots, v_n$ *do not span* K^n.
    2. $[v_1, v_2, \ldots, v_n] = [kv_1, v_2, \ldots, v_n]$ *for any nonzero* $k \in K$;
    3. $[v_1, v_2, \ldots, v_n] = (-1)^s [v_{s(1)}, v_{s(2)} \ldots, v_{s(n)}]$ *for any permutation* $s \in S_n$;
    4. $[v_1, v_2, \ldots, v_n] = [x, v_2, \ldots, v_n] + \cdots + [v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n] + \cdots + [v_1, v_2, \ldots, v_{n-1}, x]$ *for any nonzero* $x \in K^n$.

Passing Through x

We call the fourth relation “passing through x”.

$$[v_1, v_2, \ldots, v_n] = [x, v_2, \ldots, v_n] + \cdots + [v_1, \ldots, v_{i-1}, x, v_{i+1}, \ldots, v_n] \\ + \cdots + [v_1, v_2, \ldots, v_{n-1}, x]$$

for any nonzero $x \in K^n$.

Steinberg Homology

For $\Gamma \leq \mathrm{GL}_n(K)$ a congruence group, the *Steinberg homology* of Γ is

$$H_*(\Gamma, \mathrm{St}(K^n)).$$

- The Steinberg homology is stable under any Hecke operator $\Gamma s \Gamma$ for any $s \in \mathrm{GL}_n(K)$ that commensurates Γ.
- Example: $R = \mathbb{C}$, $n = 2$: $H_0(\Gamma, \mathrm{St}(\mathbb{Q}^2; \mathbb{C}))$ computes weight 2 modular forms for Γ.

Borel-Serre Duality Theorem

Theorem (Borel-Serre, 1973)

Let Γ be a subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$. For any i, there is an isomorphism

$$H_i(\Gamma, \mathrm{St}(\mathbb{Q}^n)) \simeq H^{\nu-i}(\Gamma, \mathbb{Q}),$$

where

$$\nu = \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2}.$$

The zero-th Steinberg homology group, $H_0(\Gamma, \mathrm{St}(\mathbb{Q}^n))$, will be identified with the group of co-invariants $\mathrm{St}(\mathbb{Q}^n)_\Gamma$.

A Filtration

Theorem (Ash, 2018)

There is a filtration of $\mathrm{St}(K^n)$ *that is stable under the natural action of* $\mathrm{GL}_n(\mathbb{Q})$*:*

$$0 \subsetneq \mathrm{St}(\mathbb{Q}^n) = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_n = \mathrm{St}(K^n),$$

where $\mathcal{F}_m$ *is the* $\mathbb{Q}$*-span of all modular symbols* $[a_1, \ldots, a_{n-m}, b_1, \ldots, b_m] \in \mathrm{St}(K^n)$ *where* $a_i \in \mathbb{Q}^n$ *for all* i *and* $b_j \in K^n$ *for all* j*.*

Pure Subspaces

Ash shows we can find representatives such that $W = \langle b_1, \ldots, b_m \rangle$ is *pure*, i.e., $W \cap \mathbb{Q}^n = \{0\}$.

Let $d = [K : \mathbb{Q}]$. If W is pure, then $\dim_{\mathbb{Q}}(W) + n \leq dn$. Thus

$$\dim_K(W) \leq \frac{dn - n}{d}.$$

A Short Exact Sequence

Set $n = 3$ and $K = E = \mathbb{Q}(\sqrt{\Delta})$ a real quadratic field so that $d = 2$. Then $\frac{dn-n}{d} = \frac{3}{2}$.

- (Pure subspaces): $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$, so that the filtration has only one step:

$$0 \subsetneq \mathrm{St}(\mathbb{Q}) = \mathcal{F}_0 \subsetneq \mathcal{F}_1 = \mathrm{St}(E^3).$$

- The inclusion $\mathrm{St}(\mathbb{Q}^3)$ into $\mathrm{St}(E^3)$ gives rise to an exact sequence of $\mathbb{Q}\,\mathrm{GL}_3(\mathbb{Q})$-modules

$$0 \to \mathrm{St}(\mathbb{Q}^3) \to \mathrm{St}(E^3) \to C_E \to 0.$$

A "Natural" Subspace of $H^3(\Gamma, \mathbb{Q})$

For $\Gamma \leq \mathrm{GL}_3(\mathbb{Q})$, we get a long exact sequence in homology.

Let $H(\Gamma, E)$ be the image of the connecting homomorphism,

$$H_1(\Gamma, C_E) \xrightarrow{\psi} H_0(\Gamma, \mathrm{St}(\mathbb{Q}^3)).$$

GOAL: Investigate

$$H(\Gamma, E) \subseteq H_0(\Gamma, \mathrm{St}(\mathbb{Q}^3)) \simeq \mathrm{St}(\mathbb{Q}^3)_\Gamma \simeq H^3(\Gamma, \mathbb{Q})$$

as E varies.

Tasks

1. Describe C_E explicitly as a direct sum of induced Γ-modules, $C_E \simeq \bigoplus \mathbb{I}(a, b, d)$.
2. Use 1 to describe $H_1(\Gamma, C_E)$ and ψ explicitly.
3. Use 1 and 2 to describe vectors in $H(\Gamma, E)$.
4. Compute.

Focus on 3 and 4 today.

Unital Matrices and Double Coset Representatives

- A 2×2 matrix $h \in \mathrm{GL}_2(\mathbb{Z})$ is *unital* if $h(\beta : 1) = (\beta : 1)$ for some $\beta \in E \setminus \mathbb{Q}$.
- For h unital and $u \in \mathbb{Z}^2$, let

$$M(h,u) = \begin{bmatrix} h & u \\ 0 & \epsilon \end{bmatrix} \in \mathrm{SL}_3(\mathbb{Z}).$$

  $M(h,u)$ has eigenvectors $b = {}^t(\beta, 1, 0)$, b' the Galois conjugate of b, and $a = {}^t(a_1, a_2, 1) \in \mathbb{Z}^3$.
- Let $P_3 \subseteq \mathrm{SL}_3(\mathbb{Z})$ be the stabilizer of the plane ${}^t(*, *, 0)$. Let L be (finite) a list of representatives of the double cosets $\Gamma \backslash \mathrm{SL}_3(\mathbb{Z}) / P_3$.

Image $H(\Gamma, E)$

Theorem

$H(\Gamma, E)$ is the $\mathbb{Q}$-span of all $[f, \gamma f, da]_\Gamma$, where

- *For each unital h and $u \in \mathbb{Z}^2$, form $M(h, u)$ and find eigenbasis $\{b, b', a\}$.*
- *Choose double coset representative $d \in L$.*
- *Find the smallest positive power of $dM(h, u)d^{-1}$ which lies in Γ. Call this γ.*
- *Set f to be the first column of d.*

Computational Tasks

1. Find many unital h.
2. Compute $[f, \gamma f, da]_\Gamma$ and Hecke operators in terms of unimodular symbols.
3. Represent cohomology in the computer as Voronoi homology. (Fix basis of unimodular symbols.)
4. Compute $[f, \gamma f, da]_\Gamma$ and Hecke operators in terms of unimodular symbols.

Finding Unital Matrices

A matrix h is unital if and only if $h\begin{bmatrix}\beta\\1\end{bmatrix} = \eta\begin{bmatrix}\beta\\1\end{bmatrix}$ for some unit $\eta \in \mathcal{O}_E^{\times}$.
Let the minimal polynomial of η be $f(x) = x^2 - tx + n$. Then take

$$h = \begin{bmatrix} t-d & -\frac{f(d)}{cN} \\ cN & d \end{bmatrix}.$$

Hecke Operators $T(\ell, k)$ for primes $\ell \nmid N$

$T(\ell, 0)$ and $T(\ell, 3)$ are each the identity map. Set

$$D(\ell, 1) = \mathrm{diag}(1, 1, \ell) \quad \text{and} \quad D(\ell, 2) = \mathrm{diag}(1, \ell, \ell).$$

For $k = 1, 2$, express the double coset as a disjoint union of left cosets

$$\Gamma D(\ell, k)\Gamma = \coprod_{h \in \Omega_k} \Gamma h.$$

The action of $T(\ell, k)$ on the symbol $[v_1, v_2, v_3]$ is given by

$$T(\ell, k)[v_1, v_2, v_3] = \sum_{h \in \Omega_k} [hv_1, hv_2, hv_3].$$

Passing through x

Problem: We have cohomology represented by unimodular symbols, and $[hv_1, hv_2, hv_3]$ and $[f, \gamma f, da]$ are not unimodular.

Solution: Reduction algorithm "passing through x",

$$[w_1, w_2, w_3] = [x, w_2, w_3] + [w_1, x, w_3] + [w_1, w_2, x],$$

for well-chosen nonzero vector x.

Reduction Algorithm

[van Geemen-van der Kallen-Top-Verberkmoes 1997, Ash-Rudolph 1979]

Let $A = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$ with $\det(A) > 1$.

- There exists $\overline{a}_1, \overline{a}_2, \overline{a}_3 \in \mathbb{Z}/m\mathbb{Z}$, not all zero, such that

$$\overline{a}_1 \overline{w}_1 + \overline{a}_2 \overline{w}_2 + \overline{a}_3 \overline{w}_3 = 0 \in (\mathbb{Z}/m\mathbb{Z})^3.$$

- There exists $a_1, a_2, a_3 \in \mathbb{Z}$, not all zero, such that
  - $x = \frac{1}{m}(a_1 w_1 + a_2 w_2 + a_3 w_3) \in \mathbb{Z}^3$
  - $|a_i| \leq m/2$

Reduction Algorithm

Passing through x:

$$[w_1, w_2, w_3] = [x, w_2, w_3] + [w_1, x, w_3] + [w_1, w_2, x]$$

Note:

$$|\det([x, w_2, w_3])| = \left|\frac{a_1}{m}\det([w_1, w_2, w_3])\right| \leq \frac{1}{2}\det(A).$$

Similarly,

$$|\det([w_1, x, w_3])|, |\det([w_1, w_2, x])| \leq \frac{1}{2}\det(A).$$

Passing through x shrinks the determinants.

Boundary and Interior Cohomology [Lee-Schwermer, 1982]

Let $\mathcal{S}$ be the 5-dimensional symmetric space for $\mathrm{SL}_3(\mathbb{R})$, and let X be the Borel-Serre compactification of $\Gamma\backslash\mathcal{S}$. Let $T = T(\mathbb{Q}^n)$.

- The *interior cohomology* $H^3_!(X,\mathbb{Q})$ is the kernel of the restriction map

$$H^3(X,\mathbb{Q}) \xrightarrow{\ r\ } H^3(\partial X,\mathbb{Q}) = A'(\Gamma) \oplus B'(\Gamma).$$

- $B'(\Gamma) \simeq H_1(T/\Gamma,\mathbb{Q})$ and $A'(\Gamma)$ “comes from” the maximal parabolic subgroups.

We get Hecke-equivariant decomposition

$$H^3(\Gamma,\mathbb{Q}) = H^3_!(\Gamma,\mathbb{Q}) \oplus A(\Gamma) \oplus B(\Gamma).$$

Conjecture Based on Computation

Conjecture

1. $H(\Gamma, E) = H^3_!(\Gamma, \mathbb{Q}) \oplus A(\Gamma)$
2. $\dim_{\mathbb{Q}}(H(\Gamma, E)) = \dim_{\mathbb{Q}}(H^3(\Gamma, \mathbb{Q})) - \dim_{\mathbb{Q}}(H_1(T/\Gamma, \mathbb{Q}))$

- 1 implies 2.
- 2 was first formulated based on numerical experiments: $\Gamma = \Gamma_0(N)$ with $N \leq 50$, prime $N < 100$, and $N = 11^2, 13^2$; $E = \mathbb{Q}(\sqrt{\Delta})$, $\Delta = 2, 3, 5, 6, 7, 10$.
- 1 came later after Hecke analysis.

Hecke Analysis

- Simultaneously diagonalize the Hecke operators $T(\ell, k)$ acting on $H^3(\Gamma, \mathbb{Q})$.
- For each Hecke eigenclass, form the Hecke polynomials

$$P_\ell(X) = 1 - a(\ell, 1)X + a(\ell, 2)\ell X^2 - \ell^3 X^3.$$

- Match to compatible families of Galois representations, i.e., find ρ such that

$$P_\ell(X) = \det(1 - \rho(\mathrm{Frob}_\ell)X).$$

Use Hecke Analysis to Determine Constituents

Use the attached Galois representations to classify Hecke eigenclasses [Ash-Stevens, 1986].

- $B(\Gamma) \otimes_{\mathbb{Q}} \mathbb{C}$ eigenclass: attached to a direct sum of 3 Dirichlet characters of levels dividing N.
- $A(\Gamma) \otimes_{\mathbb{Q}} \mathbb{C}$ eigenclass: attached to the direct sum of a Dirichlet character of conductor dividing N and an odd two-dimensional representation coming from a holomorphic modular form of weight 2 and level dividing N and trivial nebentype.

An Example: $N = 121$

- $\Gamma = \Gamma_0(121)$
- $H^3(\Gamma, \mathbb{Q})$ is 29-dimensional.
  - $B(\Gamma)$ is 13-dimensional.
  - $A(\Gamma)$ is 14-dimensional.
  - $H^3_!(\Gamma, \mathbb{Q})$ is 2-dimensional.
- $H(\Gamma, E)$ is 16-dimensional.

An Example: $N = 121$, $B(\Gamma)$

- Trivial (1):

$$P_\ell(X) = (1 - X)(1 - \ell X)(1 - \ell^2 X).$$

- Dirichlet characters (12): The 4 characters of order 5 in the group of Dirichlet characters of modulus 11 each contribute a 3-dimensional space

$$P_\ell(X) = (1 - X)(1 - \chi(\ell)\ell X)(1 - \chi^{-1}(\ell)\ell^2 X)$$
$$P_\ell(X) = (1 - \ell X)(1 - \chi(\ell) X)(1 - \chi^{-1}(\ell)\ell^2 X)$$
$$P_\ell(X) = (1 - \ell^2 X)(1 - \chi(\ell) X)(1 - \chi^{-1}(\ell)\ell X).$$

$B(\Gamma)$ is 13-dimensional.

An Example: $N = 121$, $A(\Gamma)$

- GL_2-newforms (8): Each of the 4 newforms $\{a_\ell\}$ at level 121 contribute a 2-dimensional space:

  $$a(\ell,1) = b(\ell,2) = a_\ell + \ell^2 \quad \text{and} \quad a(\ell,2) = b(\ell,1) = \ell a_\ell + 1$$

  so that

  $$P_\ell(X) = (1 - \ell^2 X)(1 - a_\ell X + \ell X^2)$$
  $$P_\ell(X) = (1 - X)(1 - \ell a_\ell X + \ell^3 X^2).$$

- GL_2-oldforms (6): The level 11 newform contributes as above with multiplicity 3.

$A(\Gamma)$ is 14-dimensional.

An Example $N = 121$, $H^3_!(\Gamma, \mathbb{Q})$

- Symmetric Squares (2): Let $\{a_\ell\}$ denote the GL_2-newform 121.2.a.d, 121.2.a.a, or 121.2.a.c. There is a cuspidal Hecke eigenclass in $H^3(\Gamma, \mathbb{Q})$ $\{a(\ell, k)\}$ such that

$$a(\ell, 1) = a(\ell, 2) = a_\ell^2 - \ell.$$

so that

$$P_\ell(X) = (1 - \ell X)(1 - (a_\ell^2 - 2\ell)X + \ell^2 X^2).$$

121.2.a.a and 121.2.a.c are quadratic twists of each other.

$H^3_!(\Gamma, \mathbb{Q})$ is 2-dimensional.

An Example $N = 121$, $H(\Gamma, E)$

$$\dim_{\mathbb{Q}}(H(\Gamma, E)) = 29 - 13 = 16 = 14 + 2.$$

Thank you.